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PRACTICAL LESSONS
IN
ACTUARIAL SCIENCE

AN ELEMENTARY TEXT-BOOK

SECOND EDITION
REVISED

CONTAINING ALSO QUESTIONS AND EXAMPLES

Volume I. TEXT.

BY

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INTRODUCTION.

SECOND EDITION, REVISED.

VOLUME I. TEXT.

THIS edition of PRACTICAL LESSONS IN ACTUARIAL SCIENCE has been carefully revised.

The text has been read for errors very thoroughly by two expert readers, one reading the old edition and one the copy and proofs of the new, besides reading of both by the author.

a_x has everywhere been changed to a_x in order to conform closely to the Universal Notation. All other changes deemed necessary in order to so conform have been made also, it is believed.

The final form of N_x, i. e. $= D_{x+1} + D_{x+2} + \ldots\ldots$, has been retained because still retained in the Institute of Actuaries' Text-Book, Part II., to which the student will pass from this book; but the initial form of $\mathfrak{N}_x = D_x + D_{x+1} + \ldots$ is coming into more general use even in Great Britain, in consequence of the wider employment of select tables. The student has but to remember, however that $\mathfrak{N}_x = N_{x-1}$ and the formulas become interchangeable.

Chapters upon Interest and Annuities Certain have been added: also on the following topics:

Graduation—Finlaison's Formula.
Graduation—Woolhouse's Formula.
Graduation—Makeham's Formula (amplified only).
Constructing a Select Table.
Annuities Payable Fractionally. Continuous Annuities.
Premiums Payable Fractionally.

Varying Premiums.

Insurances Payable Otherwise Than at End of Year.

Complete Annuities.

Life Annuity *versus* Annuity for Expectancy.

Net Extra Premium for Return of Premiums.

Non continuous Retrospective Valuation.

Mean and Other Ad Interim Reserves.

Reserves. Select. Select and Ultimate.

Preliminary Term Reserves.

Exposed to Risk and Expected Losses.

Joint Life Values. Equal Ages and Uniform Seniority, by Makeham's Method. (Amplified.)

Makeham's Law and Changed Rate of Interest.

Continuous Instalment Insurance. (Robertson's shorter method.)

Complex Problems in Survivorship.

Survivorship. Three Lives.

Many Unusual Annuities and Insurances.

Considerable omissions from the text of the first edition have also been made, the topics being covered in a better manner, it is hoped, in the new chapters.

Two new methods for contracted computation are given, one for multiplication, invented by Prof. Lehmer, not believed to be known to actuaries generally, and one for division, authorized to be published in this book for the first time by its inventor, Mr. Emory McClintock.

The purposes in all the revisions have been: To make the book as accurate and as simple as possible; to include everything that is strictly practical, and to exclude everything that is theoretical merely.

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PRACTICAL LESSONS IN ACTUARIAL SCIENCE.

SCOPE OF THE SCIENCE.

To define actuarial science is rendered somewhat difficult by the circumstance that not all who use the term mean by it the same thing. As used in other countries, it is so broad as to cover all kinds of expert knowledge of the use of compound interest and discount in solving financial problems, as well as a knowledge of the application of the laws of probability to insurance. Thus the formulation of plans for a building association and the distribution of its profits among its members are there regarded as actuarial functions. This comprehensive significance of the term, however, is only beginning to be recognized in the United States. Here we have in the past commonly understood the field of an actuary to be limited to technical life insurance as practiced by the regular companies. When actuaries have on occasion undertaken other financial problems, such as the distribution of building association's profits, they have been rather apologetic about it, as if it were not altogether professional. And it is true that in these days of prepared tables and ready reckoners the solution of such problems is not attended with difficulties which are insuperable to persons who are not actuaries. Eliminating this, then, we have remaining a distinctive field for actuarial science which may be defined, therefore, as the application of the laws of probability to insurance.

This definition calls for an accurate conception of insurance. Insurance is furnishing to persons, suffering a loss in a specified manner, indemnity through apportioning the loss among a number of persons who were, during a stated time, subject to the risk of a similar loss and who were covered by a similar contract of indemnity. When this apportionment is made after the losses actually occur, the operation may be simple enough, especially when the exposures are considered to involve equivalent hazards. In that case, it is only necessary to divide the loss ratably among the insured, each bearing his share. But, in practice, ordinarily, this division needs to be made before the losses occur, though in many cases subject to future revision; in other words, it is necessary to compute in advance the probability of the loss for each individual. To do this, it is required to deal with large aggregates so as to reach reliable averages, it being the primary law of probabilities that when, in a large number of cases in which an event might either happen or not happen, it does happen a certain number of times, the chance of its happening is ascertained by dividing the number of times it did happen by the total number of cases. Sometimes, these simple probabilities are very difficult to determine; and, when it is remembered that this primary problem is multiplied and involved with other more intricate problems, it appears that actuarial science has an extended and peculiar field.

Insurance is of the nature of a "hedge." We find ourselves, in the nature of things, subject to certain hazards, such as of loss of property or health or life; we cannot altogether escape these hazards, however careful we may be. We can, however, in a degree, guard against the worst consequences by "hedging." The office of a "hedge" is to cancel a hazard already incurred. But the "hedge" is necessarily of the nature of a bet; and it follows, therefore, that the same technical knowledge of probabilities which fits one for the direction of an insurance company, designed to diminish the inequalities of fortune, also fits one to direct a lottery, designed to increase those inequalities. Such employment of the skill attained by a study of the science is closely analogous to the employment of medical skill to destroy life instead of to preserve it.

Speaking absolutely, there is no such thing as chance; all phenomena are the results of causation. But from the view-point of the individual, everything is chance which he does not voluntarily cause or fail to prevent. If we could discern and compute the significance of all causes, there would be no question about probability in anything. We are able to do this to a considerable degree and one of the first tasks of the actuary is to so discriminate that, so far as he is able to determine *prima facie*, each individual in the aggregate is equally liable to the same hazard. Thus, for instance, it would be foolish to derive the probability of the death within the year of a man aged eighty, from division of the number dying by the lives exposed in a total made up of all ages; for, *prima facie*, the hazards on the individuals making up the aggregate are not the same.

The advance payment by the assured to cover his indemnity is called a premium. When the actuary has determined from a consideration of aggregates what the hazard is, a basis for this premium has been reached. Another problem at once presents itself, namely: to provide for expenses, a provision which must also be incorporated in the premium. And, finally, if the premium covers insurance for a considerable term, there enters into the calculation the matter of the interest which current balances of premium may earn, which interest is, in a very real sense, a part of the premium itself.

The adjustment of the actual costs for insurance and expense to the assumed costs after the term has expired is the next problem in order, together with the proper apportionment of the net profit or loss among the stockholders or policyholders. This is an operation often requiring the greatest nicety of the actuary's skill. And as the policies of a company are not precisely concurrent, beginning on the same day and expiring on the same day, it follows that in making this adjustment the question of a sufficient reserve to cover the unexpired contracts is directly involved.

While the three items of rate-making, surplus or loss apportionment and reserve computation comprise the major part of the mathematical phase of actuarial science, there are other intricate problems though of less moment. And, in particular, it is always assumed that an actuary is versed in all technical questions relating to life insurance, including the legal construction of contracts. The want of such knowledge will seriously impede his success. The Institute of Actuaries of Great Britain recognizes this by requiring an examination on these matters as well as insurance mathematics.

This little text-book will treat insurance mathematics only, and only so far as that can be comprehended and made use of by persons who are familiar with ordinary algebra.

INTEREST AND ANNUITIES CERTAIN.

INTEREST.

Simple Interest. When interest is not counted upon interest, when due and unpaid, it is called simple interest. If \$1000 were put out at simple interest at five per cent per annum, it would accumulate to \$1050 in one year, \$1100 in two years, etc.

Let i=rate of interest per annum.
$1+i$=accumulation of \$1.00 in one year.
v=present value of \$1 00, due in one year.
d=discount on \$1.00, due in one year.
v^n=present value of \$1.00, due in n years.
P=principal invested.
S=the accumulation of P.

$$S=P(1+ni)$$

$$P=\frac{S}{1+ni}$$

$$i=\frac{S-P}{nP}$$

$$n=\frac{S-P}{iP}$$

When the principal invested is \$1.00, these become:

$$S=1+ni$$

$$i=\frac{S-1}{n}$$

$$n=\frac{S-1}{i}$$

When the accumulation is \$1.00, they become:

$$1=P(1+ni)$$

$$P=\frac{1}{1+ni}=v^n$$

$$i = \frac{1 - P}{n P} = \frac{1 - v^n}{n v^n}$$

$$n = \frac{1 - P}{i P} = \frac{1 - v^n}{i v^n}$$

$$v = \frac{1}{1 + i}, d = 1 - v = 1 - \frac{1}{1 + i} = \frac{i}{1 + i}$$

Compound Interest. When interest is counted upon interest, when due and unpaid, it is called compound interest. If $1000 were put out at compound interest at five per cent per annum it would accumulate to $1050 in one year, to $1,102.50 in two years, etc

$$S = P (1 + i)^n$$

$$^{*}P = \frac{S}{(1 + i)^n} = S \frac{1}{(1 + i)^n} = S v^n$$

By taking logarithms, we may get the value of n, thus:

$$S = P (1 + i)^n$$

$$\log S = \log P + n \log (1 + i)$$

$$n \log (1 + i) = \log S - \log P$$

$$n = \frac{\log S - \log P}{\log (1 + i)}$$

The value of i may be found thus:

$$S = P (1 + i)^n$$

$$(1 + i)^n = \frac{S}{P}$$

$$(1 + i) = \left(\frac{S}{P}\right)^{\frac{1}{n}}$$

$$i = \left(\frac{S}{P}\right)^{\frac{1}{n}} - 1$$

This expression looks formidable, but is easily solved by putting $\left(\frac{S}{P}\right)^{\frac{1}{n}}$ into logarithms, thus : $\frac{1}{n}$ (log S—log P) and looking up the corresponding natural number.

---

[*] In simple interest $v^n = \frac{1}{1 + n i}$

In compound interest $v^n = \frac{1}{(1 + i)^n} = (1 + i)^{-n}$

When the principal is \$1.00, the foregoing become:

$$S = (1 + i)^n = \frac{1}{(1 + i)^{-n}} = \frac{1}{v^n}$$

$$n = \frac{\log S}{\log (1 + i)}$$

$$i = S^{\frac{1}{n}} - 1$$

When the accumulation is \$1.00, the equations become:

$$1 = P\,(1 + i)^n = v^n\,(1 + i)^n$$

$$P = \frac{1}{(1 + i)^n} = (1 + i)^{-n} = v^n$$

Compound Interest Due More Frequently Than Annually.

Let j = nominal rate of interest per annum.

i = effective rate of interest per annum.

If interest is due semi-annually, at the end of one year:

$$S = P\left(1 + \frac{j}{2}\right)^2$$

Or when principal is \$1.00,

$$S = \left(1 + \frac{j}{2}\right)^2 = 1 + i$$

$$\therefore i = S - 1 = \left(1 + \frac{j}{2}\right)^2 - 1$$

$$j = 2\left\{(1 + i)^{\frac{1}{2}} - 1\right\}$$

When accumulation is \$1.00.

$$1 = P\left(1 + \frac{j}{2}\right)^2$$

$$P = \frac{1}{\left(1 + \frac{j}{2}\right)^2} = \left(1 + \frac{j}{2}\right)^{-2}$$

For n years, we have:

$$S = P\left(1 + \frac{j}{2}\right)^{2n}$$

$$P = \frac{S}{\left(1 + \frac{j}{2}\right)^{2n}}$$

$$n = \frac{\log S - \log P}{2 \log\left(1 + \frac{j}{2}\right)}$$

$$j = 2 \left\{ \left(\frac{S}{P}\right)^{\frac{1}{2n}} - 1 \right\}$$

When interest is due m times a year,

$$S = P\left(1 + \frac{j}{m}\right)^{mn}$$

$$P = \frac{S}{\left(1 + \frac{j}{m}\right)^{mn}}$$

$$n = \frac{\log S - P}{m \log\left(1 + \frac{j}{m}\right)}$$

$$j = m \left\{ \left(\frac{S}{P}\right)^{\frac{1}{mn}} - 1 \right\}$$

Or, if the term is one year, and the principal is $1.00,

$$S = \left(1 + \frac{j}{m}\right)^{m} = 1 + i$$

$$j = m \left\{ S^{\frac{1}{m}} - 1 \right\} = m \left\{ (1 + i)^{\frac{1}{m}} - 1 \right\}$$

$$i = \left(1 + \frac{j}{m}\right)^{m} - 1$$

Or, if the accumulation is $1.00,

$$1 = P\left(1 + \frac{j}{m}\right)^{m} = P\,(1 + i)$$

$$P = \frac{1}{\left(1 + \frac{j}{m}\right)^{m}} = \left(1 + \frac{j}{m}\right)^{-m}$$

Compound Interest Due Momently. We may conceive of interest as accruing and compounding momently; and by means of the logarithmic series $e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x}{\underline{|3}} + \ldots$ we may get an expression for its values. Let m become indefinitely great in the formula:

$$S = \left(1 + \frac{j}{m}\right)^m = 1 + i$$

And we have

$$S = e^j = 1 + i$$

In such cases j = nominal rate per annum, payable momently. In order to distinguish, let δ = such a rate and we have:

$$e^\delta = 1 + i$$

$$\therefore\ \delta = \log_e (1 + i)$$

Making m indefinitely great in

$$S = P\left(1 + \frac{j}{m}\right)^{mn}$$

we have $S = Pe^{\delta n}$

$$n = \frac{\log S - \log P}{\delta}$$

Making m indefinitely great in

$$v = \frac{1}{1 + i} = \frac{1}{\left(1 + \frac{j}{m}\right)^m} = \left(1 + \frac{j}{m}\right)^{-m}$$

we have $v = e^{-j} = 1 - d$

and substituting δ for j, we have

$$v = e^{-\delta} = 1 - d$$

$$\therefore -\delta = \log_e v = \log_e (1 - d)$$

$$\delta = -\log_e (1 - d)$$

Making m indefinitely great in

$$P = \frac{S}{\left(1+\frac{j}{m}\right)^{mn}} = S\left(1+\frac{j}{m}\right)^{-mn}$$

we have: $P = Se^{-jn}$

and substituting δ for j,

$$P = Se^{-\delta n}$$

Expanding the two equations for the value of δ in terms of i and d, we have:

$$\delta = \log_e (1+i) = i - \frac{i^2}{2} + \frac{i^3}{3} - \frac{i^4}{4} + \ldots\ldots$$

$$\delta = -\log_e (1-d) = d + \frac{d^2}{2} + \frac{d^3}{3} + \frac{d^4}{4} + \ldots$$

$$\therefore\ 2\delta = i + d - \left(\frac{i^2-d^2}{2}\right) + \frac{i^3+d^3}{3} - \left(\frac{i^4-d^4}{4}\right) + \ldots$$

$$\delta = \frac{i + d - \left(\frac{i^2-d^2}{2}\right) + \frac{i^3+d^3}{3} - \left(\frac{i^4-d^4}{4}\right) + \ldots}{2}$$

$$= \frac{i+d}{2} \text{ approximately.}$$

Since $\delta = j$ when $m = \infty$, we may also derive δ from the formula

$$j = m\left\{(1+i)^{\frac{1}{m}} - 1\right\}$$

by letting $m = \infty$

Money Doubles Itself in How Many Years, at Compound Interest? **Money doubles itself when**

$$(1+i)^n = 2$$

$$\therefore\ n \log (1+i) = \log 2$$

$$n = \frac{\log 2}{\log (1+i)} = \frac{\log_e 2}{\delta}$$

$$= \frac{.30103\ (2.3026)}{\delta}$$

$$= \frac{.69315}{i - \frac{i^2}{2} + \frac{i^3}{3} - \quad . \quad . \quad . \quad .}$$

$$= \frac{.69315}{i}\left(1 + \frac{i}{2} - \frac{i^2}{12} + \quad . \quad . \quad \right)$$

$$= \frac{.69315}{i} + .35, \text{ approximately.}$$

The usual rough rule is: Divide .69 by the rate of interest.

ANNUITIES CERTAIN.

At Simple Interest. Annuities certain, valued on the principle of simple interest, have only a theoretical interest for the student and will be treated very briefly.

Let $a_{\overline{n}|}$ = present value of an annuity of \$1.00, payable at the end of each year for n years.

$\mathbf{a}_{\overline{n}|}$ = present value of an annuity of \$1.00, payable at the beginning of each year for n years.

$s_{\overline{n}|}$ = accumulated value of an annuity of \$1 00, payable at the end of each year for n years, but not drawn until the end of n years.

a_{∞} = present value of a perpetuity of \$1.00 at the end of each year.

If one were to put \$1.00 out at interest at rate i per annum, interest payable forever, we should have:

$$i\, a_{\infty} = 1$$

$$\therefore a_{\infty} = \frac{1}{i}$$

The accumulated value at simple interest of an annuity of \$1.00, payable at the end of each year, is expressed thus:

$$s_{\overline{n}|} = 1 + (1 + i) + (1 + 2\,i) + \;\ldots\ldots\; (1 + \overline{n - 1}.i)$$

$$= n + i + 2\,i + \;\ldots\ldots\ldots\ldots\; (\overline{n - 1}.i)$$

$$= n + \frac{n\,i}{2}(n - 1) = n + \frac{n\,(n - 1)}{2}i = \frac{n}{2}\left(2 + \overline{n - 1}.i\right).$$

$$a_{\overline{n}|} = \frac{1}{1 + i} + \frac{1}{1 + 2\,i} + \frac{1}{1 + 3\,i} + \;\ldots\ldots\; \frac{1}{1 + \overline{n - 1}.i}$$

For this latter expression no accurate summation by algebraic formula has been discovered. Three rough approximations have been suggested as follows:

1. Accumulate the annuity for n years and take the present value of the accumulation; that is:

$$a_{\overline{n|}} = \frac{s_{\overline{n|}}}{1 + n i} = \frac{n\,(2 + \overline{n - 1}.i)}{2\,(1 + n i)}$$

2. Take the present value of a perpetuity less the present value of a perpetuity deferred n years:

$$a_{\overline{n|}} = a_{\infty} - {}_{n|}a_{\infty} = \frac{1}{i} - \frac{1}{i} \cdot \frac{1}{1 + n i} = \frac{1}{i}\left(1 - \frac{1}{1 + n i}\right)$$

$$= \frac{1}{i} \cdot \frac{n i}{1 + n i} = \frac{n}{1 + n i}$$

3. For the present value, take the accumulated value less the accumulated value of the interest on the present value, thus:

$$a_{\overline{n|}} = s_{\overline{n|}} - i\, a_{\overline{n|}}\, s_{\overline{n|}}$$

$$a_{\overline{n|}}\,(1 + i\, s_{\overline{n|}}) = s_{\overline{n|}}$$

$$a_{\overline{n|}} = \frac{s_{\overline{n|}}}{1 + i\, s_{\overline{n|}}}$$

At Compound Interest. The valuation of annuities certain on the principle of compound interest, is of great practical importance. Most of the valuable functions have been tabulated, but the principles and rules of computation form an important branch of actuarial science.

1. Interest payable annually:

Annuity payable once a year.

$$s_{\overline{n|}} = 1 + (1 + i) + (1 + i)^2 + \;.\;\;.\;\;.\;\;.\;\; (1 + i)^{n-1}$$

$$= \frac{(1 + i)^n - 1}{i}$$

$$a_{\overline{n|}} = v + v^2 + v^3 + \;.\;\;.\;\;.\;\;.\;\; v^n$$

$$= \frac{1 - v^n}{i}$$

$$\mathbf{a}_{\overline{n|}} = 1 + a_{\overline{n-1|}}$$

Annuity, $\frac{1}{p}$ at the end of each 1/pth part of the year:

$$s^{(p)}_{\overline{n|}} = \frac{1}{p} + \frac{1}{p}(1+i)^{\frac{1}{p}} + \frac{1}{p}(1+i)^{\frac{2}{p}} + \dots \frac{1}{p}(1+i)^{n-\frac{1}{p}}$$

$$= \frac{1}{p}\left\{1 + (1+i)^{\frac{1}{p}} + (1+i)^{\frac{2}{p}} + \dots (1+i)^{n-\frac{1}{p}}\right\}$$

$$= \frac{1}{p}\cdot\frac{(1+i)^n - 1}{(1+i)^{\frac{1}{p}} - 1} = \frac{(1+i)^n - 1}{p\left\{(1+i)^{\frac{1}{p}} - 1\right\}}$$

$$a^{(p)}_{\overline{n|}} = \frac{1}{p}v^{\frac{1}{p}} + \frac{1}{p}v^{\frac{2}{p}} + \dots \frac{1}{p}v^n$$

$$= \frac{1}{p}\left(v^{\frac{1}{p}} + v^{\frac{2}{p}} + v^{\frac{3}{p}} + \dots v^n\right)$$

$$= \frac{1}{p}\cdot\frac{1 - v^n}{(1+i)^{\frac{1}{p}} - 1} = \frac{1 - v^n}{p\left\{(1+i)^{\frac{1}{p}} - 1\right\}}$$

$$\mathbf{a}^{(p)}_{\overline{n|}} = \frac{1}{p} + a^{(p)}_{\overline{n-\frac{1}{p}|}}$$

Annuity accruing momently: In the last formulas, let $p = \infty$ and we have.

$$\bar{s}_{\overline{n|}} = \frac{(1+i)^n - 1}{\delta}$$

$$\bar{a}_{\overline{n|}} = \frac{1 - v^n}{\delta}$$

since δ is the symbol for the nominal rate j, payable m times a year, when $m = \infty$ and since

$$j = m\left\{(1+i)^{\frac{1}{m}} - 1\right\}$$

2. Interest payable each 1/m th part of year; replace $(1 + i)^n$ and v^n by their equivalents.

Annuities payable annually:

$$s_{\overline{n}|} = \frac{\left(1+\frac{j}{m}\right)^{mn}-1}{\left(1+\frac{j}{m}\right)^{m}-1}$$

$$a_{\overline{n}|} = \frac{1-\left(1+\frac{j}{m}\right)^{-mn}}{\left(1+\frac{j}{m}\right)^{m}-1}$$

Annuities payable $\frac{1}{p}$ every $\frac{1}{p}$ th of year:

$$s^{(p)}_{\overline{n}|} = \frac{\left(1+\frac{j}{m}\right)^{mn}-1}{p\left\{\left(1+\frac{j}{m}\right)^{\frac{m}{p}}-1\right\}}$$

$$a^{(p)}_{\overline{n}|} = \frac{1-\left(1+\frac{j}{m}\right)^{-mn}}{p\left\{\left(1+\frac{j}{m}\right)^{\frac{m}{p}}-1\right\}}$$

Annuities payable $\frac{1}{m}$ every $\frac{1}{m}$ th year:

$$s^{(m)}_{\overline{n}|} = \frac{\left(1+\frac{j}{m}\right)^{mn}-1}{j}$$

$$a^{(m)}_{\overline{n}|} = \frac{1-\left(1+\frac{j}{m}\right)^{-mn}}{j}$$

Annuities accruing momently:

$$\bar{s}_{\overline{n}|} = \frac{\left(1+\frac{j}{m}\right)^{mn}-1}{m\log_e\left(1+\frac{j}{m}\right)}$$

$$\bar{a}_{\overline{n}|} = \frac{\left(1 + \frac{j}{m}\right)^{mn} - 1}{m \log_e \left(1 + \frac{j}{m}\right)}$$

3. At interest compounded momently:

Annuities payable annually:

$$s_{\overline{n}|} = \frac{e^{\delta n} - 1}{e^{\delta} - 1}$$

$$a_{\overline{n}|} = \frac{1 - e^{-\delta n}}{e^{\delta} - 1}$$

Annuities payable $\frac{1}{p}$ each $\frac{1}{p}$th year:

$$s_{\overline{n}|}^{(p)} = \frac{e^{\delta n} - 1}{p\left(e^{\frac{\delta}{p}} - 1\right)}$$

$$a_{\overline{n}|}^{(p)} = \frac{1 - e^{-\delta n}}{p\left(e^{\overset{\delta}{p}} - 1\right)}$$

Annuities accruing momently:

$$\bar{s}_{\overline{n}|} = \frac{e^{\delta n} - 1}{\delta}$$

$$\bar{a}_{\overline{n}|} = \frac{1 - e^{-\delta n}}{\delta}$$

Perpetuities. The reasoning which applied as to perpetuities in simple interest, applies also when interest is compounded. That is, if \$1 00 is invested permanently at interest payable annually at rate i, we have:

$$i\,a_{\infty} = 1$$

$$\therefore\ a_{\infty} = \frac{1}{i}$$

Or, if at interest payable each $\frac{1}{p}$th part of year at annual rate j, then:

$$j\,a_{\infty}^{(p)} = 1$$

$$\therefore \quad a^{(p)}_{\infty} = \frac{1}{j}$$

Or, if at interest payable momently (or compounding momently) at annual rate δ, then:

$$\delta \bar{a}_{\infty} = 1$$

$$\therefore \quad \bar{a}_{\infty} = \frac{1}{\delta}$$

Annuity and Perpetuity Increasing in Arithmetical Progression.

$$a = v\,p + v^2(p+q) + v^3(p+2q) + \ldots\; v^n(p+\overline{n-1}.q)$$

$$v\,a = v^2p + v^3(p+q) + \ldots\; v^n(p+\overline{n-2}.q) + v^{n+1}(p+\overline{n-1}.q)$$

$$\therefore (1-v)\,a = v\,p + v^2q + v^3q + v^4q + \ldots\; v^n\,q - v^{n+1}(p+\overline{n-1}.q)$$

$$= v\,p + v\,q\,(v + v^2 + \ldots\; v^n) - v^{n+1}\,(p + n\,q)$$

$$= v\,p\,(1 - v^n) + v\,q\,a_{\overline{n}|} - n\,v^{n+1}\,q$$

$$\therefore \quad a = p\,\frac{1-v^n}{i} + q\,\frac{a_{\overline{n}|} - n\,v^n}{i} = p\,a_{\overline{n}|} + q\,\frac{a_{\overline{n}|} - n\,v^n}{i}$$

For a perpetuity, let $n = \infty$ and we have.

$$a = \frac{p}{i} + \frac{q}{i^2}$$

Therefore, when the annuity is \$1.00, increasing \$1.00 each year, we have:

$$(I\,a)_{\overline{n}|} = a_{\overline{n}|} + \frac{a_{\overline{n}|} - n\,v^n}{i}$$

$$(I\,a)_{\infty} = \frac{1}{i} + \frac{1}{i^2}$$

Annuity and Perpetuity Increasing in Geometrical Progression.

$$a = k\,v + k\,\tau\,v^2 + k\,\tau^2\,v^3 + \ldots\; k\,\tau^{n-1}\,v^n$$

$$\tau\,v\,a = k\,\tau\,v^2 + k\,\tau^2\,v^3 + \ldots\; k\,\tau^{n-1}\,v^n + k\,\tau^n\,v^{n+1}$$

$$\therefore (1-\tau\,v)\,a = k\,v - k\,\tau^n\,v^{n+1} = k\,v\,(1-\tau^n\,v^n)$$

$$a = k\,v\,\frac{1-\tau^n\,v^n}{1-\tau\,v} = k\,v\,\frac{1-\tau^n\,v^n}{1-\tau\left(\frac{1}{1+i}\right)} = k\,\frac{1-\tau^n\,v^n}{1+i-\tau}$$

If $\tau < 1+i$, the value of the perpetuity wll be $k \frac{1}{1+i-\tau} = \frac{k}{1+i-\tau}$; if $\tau > 1+i$, the value of the perpetuity will be infinitely large.

Important Observations. Annuities payable p times a year may be conveniently valued in one of the following ways, viz.:

1. Compute the accumulation of the annuity for one year at the nominal annual rate, thus:

$$s^{(p)}_{\overline{1}|} = \frac{1}{p} s'_{\overline{p}|} \text{ where } s'_{\overline{p}|} \text{ is at annual rate, } \frac{j}{p}.$$

$$i = \left(1 + \frac{j}{p}\right)^p - 1$$

$$a^{(p)}_{\overline{n}|} = a_{\overline{n}|} \cdot s^{(p)}_{\overline{1}|}$$

$$s^{(p)}_{\overline{n}|} = s_{\overline{n}|} \cdot s^{(p)}_{\overline{1}|}$$

2. Compute the value of an annuity of $1.00 for n p years at annual rate, $\frac{j}{p}$, and divide by p, thus:

$$a^{(p)}_{\overline{n}|} = \frac{1}{p} a'_{\overline{np}|} \text{ where } a'_{\overline{np}|} \text{ is at rate, } \frac{j}{p}.$$

$$s^{(p)}_{\overline{n}|} = \frac{1}{p} s'_{\overline{np}|} \text{ where } s'_{\overline{np}|} \text{ is at rate, } \frac{j}{p}.$$

The present value of an annuity being known, the accumulated value may be derived from it, or, the accumulated value being known, the present value may be derived by the following formulas:

1. Interest payable annually:

$$a_{\overline{n}|} = v^n s_{\overline{n}|}$$

$$s_{\overline{n}|} = (1+i)^n a_{\overline{n}|}$$

2. Interest payable m times a year:

$$a_{\overline{n|}} = \left(1 + \frac{j}{m}\right)^{-mn} s_{\overline{n|}}$$

$$s_{\overline{n|}} = \left(1 + \frac{j}{m}\right)^{mn} a_{\overline{n|}}$$

3. Interest compounding momently:

$$a_{\overline{n|}} = e^{-\delta n}\, s_{\overline{n|}}$$

$$s_{\overline{n|}} = e^{\delta n}\, a_{\overline{n|}}$$

To Find the Rate of Interest of an Annuity Certain. Given a table of present and accumulated values of annuities, at small differences of interest rates as for every ⅛ of one per cent, the rate of interest at which an annuity is valued may be found approximately from the amount of the value, thus:

Let $a_{\overline{n|}}$ = value of annuity for each $1.00 of annuity.

$a'_{\overline{n|}}$ at rate i' = value next greater than $a_{\overline{n|}}$ in table.

$a''_{\overline{n|}}$ at rate i'' = value next smaller than $a_{\overline{n|}}$ in table.

Then,

$$i - i' : i'' - i' :: a_{\overline{n|}} - a'_{\overline{n|}} : a''_{\overline{n|}} - a'_{\overline{n|}}$$

$$\therefore \quad \frac{i - i'}{i'' - i'} = \frac{a_{\overline{n|}} - a'_{\overline{n|}}}{a''_{\overline{n|}} - a'_{\overline{n|}}}$$

$$(i - i')\,(a''_{\overline{n|}} - a'_{\overline{n|}}) = (i'' - i')\,(a_{\overline{n|}} - a'_{\overline{n|}})$$

$$i\,(a''_{\overline{n|}} - a'_{\overline{n|}}) = i'(a''_{\overline{n|}} - a'_{\overline{n|}}) + (i'' - i')\,(a_{\overline{n|}} - a'_{\overline{n|}})$$

$$i = i' + (i'' - i')\,\frac{a_{\overline{n|}} - a'_{\overline{n|}}}{a''_{\overline{n|}} - a'_{\overline{n|}}}$$

BONDS, DEBENTURES, INSTALMENT LOANS.

Bonds and Debentures. It requires no proof that if $1000 is lent for ten years at interest at five per cent per annum, the value of the loan at the outset, computed at the same rate of interest, is $1000; but it is not so manifest, unless one stops to think, that this value is composed of the present value of $1000 due in ten years and the present value of an annuity certain of $50 for ten years.

Let $C =$ principal.
$K =$ present value of principal.
$A =$ total present value.
$g =$ rate of interest paid by loan.
$i =$ rate of interest used in valuation.

$$\begin{aligned} A &= C\,v^n + g\,C\,a_{\overline{n|}} \\ &= K + g\,C\,a_{\overline{n|}} \\ &= K + g\,C\,\frac{1-v^n}{i} \\ &= K + \frac{g}{i}\,(C-K) \end{aligned}$$

When the rate of interest used in valuation is the same as the rate paid by the loan, this equation becomes:

$$A = K + \frac{i}{i}\,(C-K) = C$$

That is, the value equals the principal of the loan.

We may consider the value, when the rate paid on the loan and the rate used in valuation are not the same, from another standpoint, viz.: The value is composed of three parts: 1st, the present value of the principal; 2nd, the present value of the interest on the principal at the rate used in valuation, these two values equalling the principal;

and, 3rd, the present value of the excess of the interest at the rate earned over the rate used in valuation or, if less, then a negative quantity equal to the present value of the deficiency; thus:

$$A = C + (g-i)\, a_{\overline{n|}}$$

Let k = premium (or discount below par if negative.)

$$k = (g-i)\, a_{\overline{n|}}$$

$$= (g-i)\, \frac{1 - v^n}{1}$$

The foregoing formulas will apply also to value bonds or debentures with interest payable more frequently than annually, provided the interest used in valuation is considered to be payable at the same intervals. Thus, if semi-annually

$$A = C\, v^{2n} + \frac{g}{2}\, C\, a_{\overline{2n|}}$$

in which v^{2n} and $a_{\overline{2n|}}$ are computed at rate $\frac{j}{2}$ when rate j per annum, payable semi-annually, is used in valuation. And, when payable quarterly, we have:

$$A = C\, v^{4n} + \frac{g}{4} C\, a_{\overline{4n|}}$$

at rate $\frac{j}{4}$ and, generally,

$$A = C\, v^{pn} + \frac{g}{p} C\, a_{\overline{pn|}}$$

at rate $\frac{j}{p}$

This general expression may also be given a form, showing the amount of premium or discount below par in the valuation, thus:

$$A = C + \frac{g-j}{p}\, a_{\overline{pn|}} = C + k$$

If interest is payable more frequently than annually and the valuation is by interest payable annually, then the accumulated value at the end of one year, computed at the rate of valuation, should be substituted for the interest at the nominal rate, thus:

$$\begin{aligned} A &= C\,v^n + g\,s_{\overline{1}|}^{(p)}\,C\,a_{\overline{n}|} \\ &= K + g\,s_{\overline{1}|}^{(p)}\,C\,\frac{1-v^n}{i} \\ &= K + \frac{g\,s_{\overline{1}|}^{(p)}}{i}\,(C-K) \end{aligned}$$

This may also be so stated as to set forth the value of the premium or discount from par, thus:

$$A = C + (g\,s_{\overline{1}|}^{(p)} - i)\,a_{\overline{n}|} = C + k.$$

All of the foregoing formulas will also apply to loans which do not mature but are redeemable at the option of the borrower at the end of n years, provided the rate of interest used in the valuation is not less than the rate paid on the loan; for in such case it is assumed that the borrower will redeem. If the rate used in valuation is higher than the rate paid, it must be assumed that the borrower will not redeem and the value becomes that of a perpetuity of the interest, viz.:

If interest is annual

$$A = \frac{gC}{i}$$

If p times a year:

$$A = \frac{g\,s_{\overline{1}|}^{(p)}\,C}{i}$$

Instalment Loans. There are two forms of loans repayable by instalments, viz.: Loans with the principal repayable by equal periodical instalments, the interest being adjusted at the end of each period to the principal yet outstanding; and loans repayable principal and interest, by equal periodical instalments.

Under the first of these, each total payment made n times a year is made up as follows, for the mth payment:

$$\text{On principal } \frac{P}{n}$$

$$\text{Interest} \quad i\left\{P-\overline{m-1}\,\frac{P}{n}\right\}=i\,\frac{(n-m+1)\,P}{n}$$

In this expression $\frac{(n-m+1)P}{n}$ is the outstanding principal just before the mth payment is made.

The total payment diminishes each year by $i.\frac{P}{n}$.

The principal of a loan, repayable principal and interest by equal periodical instalments, is not diminished each year by an equal amount; since plainly less will be required out of each payment than out of the previous one, to cover the interest—because the outstanding principal is less—and consequently, more is left to be applied to reduce the principal.

Suppose a loan of the amount, a_n, repayable by an instalment of $1.00 at the end of each year for n years. At the outset, the value of the instalments would just balance the debt, as they should.

But after the first payment is made at the end of the year, the value of the remaining payments becomes $a_{\overline{n-1|}}$, instead of $a_{\overline{n|}}$; and that sum must be the outstanding principal.

Or we may get it another way which will show how much of the first payment is applied on the principal, viz.: At the end of the first year, the accrued interest is $i\,a_{\overline{n}|}$ and the amount applied on principal is, therefore :

$$1-i\,a_{\overline{n}|} = 1-i\frac{1-v^n}{i} = 1-(1-v^n) = v^n$$

This we may readily verify by applying this reduction upon the principal, thus:

$$a_{\overline{n}|} - v^n = (v + v^2 + v^3 + \dots v^n) - v^n$$
$$= v + v^2 + v^3 + \dots v^{n-1} = a_{\overline{n-1}|}$$

It is easily shown that the next instalment is composed of $i\,a_{\overline{n-1}|}$ interest and v^{n-1} to apply on principal; and the expressions for the apportionment of the mth instalment become: $i\,a_{\overline{n-m+1}|}$ interest and v^{n-m+1} to apply in principal.

Of course, since the sum of the two is $1.00, one may be found by deducting the other from $1 00.

When a loan of $a^{(p)}_{\overline{n}|}$ is repayable by instalments of $\frac{1}{p}$ at $\frac{1}{p}$th intervals of a year, the payments may be separated as follows:

If nominal annual rate of interest, j, payable p times a year, is employed, find the division of payments of $1.00 on a debt of $p\,a^{(p)}_{\overline{n}|}$ for p n years at annual rate $\frac{j}{p}$ and divide each by p.

If effective annual rate of interest, i, is employed, we have: Interest involved in first payment: $\left\{(1+i)^{\frac{1}{p}}-1\right\} a^{(p)}_{\overline{n}|}$; wherefore the amount applied to reduce the principal is $\frac{1}{p}-\left\{(1+i)^{\frac{1}{p}}-1\right\} a^{(p)}_{\overline{n}|}$.

The expression for the interest simplifies as follows:

$$\left\{(1+i)^{\frac{1}{p}}-1\right\} a^{(p)}_{\overline{n}|} = \left\{(1+i)^{\frac{1}{p}}-1\right\}\frac{1-V_n}{p\left\{(1+i)^{\frac{1}{p}}-1\right\}} = \frac{1-V_n}{p}$$

$\therefore$ The amount to be applied on principal is:

$$\frac{1}{p} - \frac{1-v^n}{p} = \frac{v^n}{p}$$

And the reduced principal

$$a^{(p)}_{\overline{n}|} - \frac{v^n}{p} = \frac{1}{p}\left\{ v^{\frac{1}{p}} + v^{\frac{2}{p}} + \quad . \quad . \quad . \quad v^n \right\} - \frac{v^n}{p}$$

$$= \frac{1}{p}\left\{ v^{\frac{1}{p}} + v^{\frac{2}{p}} + \quad . \quad . \quad . \quad v^{n-\frac{1}{p}} \right\}$$

$$= a^{(p)}_{\overline{n-\frac{1}{p}}|}$$

It will be found, then, that the reduction of the principal is $\frac{v^n}{p}$, $\frac{v^{n-\frac{1}{p}}}{p}$, etc.; the interest is $\frac{1}{p}$, less such reduction; the outstanding principal is $a^{(p)}_{\overline{n-\frac{1}{p}}|}$, $a^{(p)}_{\overline{n-\frac{2}{p}}|}$, etc.; and the interest, computed in the principal outstanding, is $\left\{ (1+i)^{\frac{1}{p}} - 1 \right\}$ multiplied into $a^{(p)}_{\overline{n}|}$, $a^{(p)}_{\overline{n-\frac{1}{p}}|}$, etc.

The loan, repayable principal and interest by equal periodical instalments, may, however, be considered in two other ways which bring out by equivalent mathematical methods the same result and have the advantage sometimes to be more convenient.

Thus, first, the loan may be conceived to be separated into two things, viz.: A straight loan of the same amount repayable at the end of the period and bearing interest at the agreed rate, and an instalment periodically to be invested in a sinking-fund which, progressed at the same rate of interest, will at the end of the period pay off the loan.

That is, if a loan of $a_{\overline{n}|}$ be made, at interest i, at an annual instalment of $1.00 for n years, in payment of principal and interest, we have:

$$i\,a_{\overline{n}|} + \frac{a_{\overline{n}|}}{s_{\overline{n}|}} = 1$$

By reduction, this could be proved to be an identity.

It follows that we can find the amount of principal repaid, at any time, by accumulating the instalment of the sinking-fund, i. e., the amount of the loan divided by the accumulation of an annuity of $1.00 for the period of the loan. Thus the principal repaid in m years is

$$s_{\overline{m}|} \cdot \frac{a_{\overline{n}|}}{s_{\overline{n}|}} = a_{\overline{n}|}\,\frac{s_{\overline{m}|}}{s_{\overline{n}|}}$$

when $a_{\overline{n}|}$ is the amount of the loan.

The equivalence of the sinking-fund formulas to the others depends upon the same rate of interest being accumulated in the sinking-fund as is paid by the loan. If this fails, the equivalence is at an end. If the sinking-fund is accumulated at a lower rate than the loan bears, the equivalence is to an instalment loan at higher than the nominal rate of interest; if it accumulates at a higher rate, the equivalence is to an instalment loan at a lower rate than the nominal.

And, second, the loan may be conceived as accumulating at the agreed rate of interest throughout the term, no interest being paid in the meantime, and the whole periodical payment to be a sinking-fund deposit which, improved at the same rate of interest, will accumulate to just enough at the end of the period to pay off the loan with its accumulation of interest. By this aspect, we find

$$a_{\overline{n}|}\,(1 + i)^n = s_{\overline{n}|}$$

a relation which we have derived before; and on the basis of this proposition we may at any time find the principal outstanding by accumulating the loan to date and the payments to date and deducting the latter from the former.

Thus, by the three systems we have:

$$\text{Unpaid principal} = a_{\overline{n-m}|}$$
$$= a_{\overline{n}|} - s_{\overline{m}|}\,\frac{a_{\overline{n}|}}{s_{\overline{n}|}}$$
$$= a_{\overline{n}|}\,(1+i)^m - s_{\overline{m}|}$$

If now the loan be $1.00, instead of the payment, we may find the amount of the payment by either of the following:

$$\text{Payment} = \frac{1}{a_{\overline{n}|}}$$
$$= \frac{1}{s_{\overline{n}|}} + i$$
$$= \frac{(1+i)^n}{s_{\overline{n}|}}$$

(a) If then a lender desires to realize any rate of interest, i′, on his net actual investment from time to time, the regular annual instalment for n years to cover both interest and principal will be for each $1.00 lent:

$$= \frac{1}{a'_{\overline{n}|}} = \frac{1}{s'_{\overline{n}|}} + i'$$

in which $a'_{\overline{n}|}$ and $s'_{\overline{n}|}$ are at rate i′.

(b) But if he wishes to realize i′ on the whole amount originally lent, for n years, the reinvestment being at rate i, only, then the payment for each $1.00 lent:

$$= \frac{1}{a_{\overline{n}|}} + (i' - i) = \frac{1}{s_{\overline{n}|}} + i'$$

in which $a_{\overline{n}|}$ and $s_{\overline{n}|}$ are at rate i.

(c) And if he wishes to have his money accumulate for n years at rate i′—not merely the whole principal but also the interest earning that rate—while i is the rate at which reinvestment can really be made; then the payment for each $1.00 lent

$$= \frac{v^n (1 + i')^n}{a_{\overline{n}|}}$$

in which v^n and $a_{\overline{n}|}$ are at rate i.

These three cases should be sharply distinguished by the student.

Suppose that an annuity of $1.00, for n years, worth $a_{\overline{n}|}$ at rate i and $a'_{\overline{n}|}$ at rate i', had been sold on such a basis that it would pay rate i' throughout the n years on the whole capital originally invested, while reinvestment of the sums repaid could be made at rate i only. Let the value of the annuity on this basis be $a^{(i+i')}_{\overline{n}|}$; and we have from (b) in the foregoing:

$$\frac{1}{a^{(i+i')}_{\overline{n}|}} = \frac{1}{s_{\overline{n}|}} + i'$$

$$\therefore\ a^{(i+i')}_{\overline{n}|} = \frac{s_{\overline{n}|}}{1 + i' s_{\overline{n}|}}$$

Such a purchase having been made, suppose that a release of the annuity were made after m years:

1st. At the request of the payer. Then plainly he would pay the present value at rate i, since the payee would, by the hypothesis, only be able to secure that rate on reinvestment of capital. Therefore, the sum payable in redemption will be $a_{\overline{n-m}|}$.

2nd. At the request of the payee. The payee has invested $a^{(i+i')}_{\overline{n}|}$. If he now receives interest on this at rate i', and only allows interest at rate i on the remainder of the payments treated as a sinking-fund, he will be even to date. He must consider the money worth at least rate i' to him for the remainder of the term; else he would not desire a settlement. Therefore, no remuneration for future loss of interest need be considered, and the amount payable in redemption will be $a_{\overline{n|}}^{(i+i')} - (1 - i\, a_{\overline{n|}}^{(i+i')}) s_{\overline{m|}}$.

3rd. At mutual desire. In such case it may be considered that each should put the other in similar position to that he occupied before the annuity was entered upon; that is, value the remaining annuity in the same manner as the value was then found. The amount payable in redemption, then, is $a_{\overline{n-m|}}^{(i+i')}$.

To Ascertain the Rate of Interest Earned by an Instalment Loan. The amount lent, to be repaid by equal periodical instalments, covering interest and principal, is the present value of an annuity of the instalments at the rate of interest which the loan pays.

Therefore in the formulas for ascertaining the rate of interest yielded by an annuity, being the last formulas in the last chapter, let $a_{\overline{n|}}$ = the amount of the loan for each $1.00 of instalment and proceed to solve.

RUDIMENTS OF PROBABILITIES.

THE science of probabilities includes so much more than is necessary as a basis for applied actuarial science, that a knowledge of the mere rudiments of probabilities is all that the actuarial student needs. To be sure, he cannot have too much and, if he is prepared for it and has the required mathematical training, he will do well to make a more extended study of the discoveries in the realm of this science. In this elementary treatise, we can only treat of the simpler problems, a knowledge of which is essential.

The science of probabilities assumes the individual as a microcosm to reproduce the characteristics of the group as a macrocosm. We have already seen that this assumption is not really true and that if, for instance, we could accurately calculate the causes at work, it would be foreseen that of a certain group just the persons who actually die within a year were the only ones who were likely to die and that all others were certain to survive. But we cannot foresee this; and we can, by deducing averages from past experience, forecast with some accuracy how many of the whole group will die. Therefore, having selected as carefully as possible a group in which the causes known to be at work apply equally to each individual. we say that the probability of the death of each is represented by the fraction, formed by dividing the number of actual deaths in such a group by the number of lives exposed in the group.

Stated as a bald proposition in the science of probabilities, this is: If an event may happen in 10 ways and not happen in 990 ways, the probability of its happening is represented by a fraction, formed by dividing the ways it may happen

(10) by the sum of the ways it may happen and the ways it may not happen (10 + 990), or

$$\frac{10}{10 + 990} = \frac{10}{1000} = \frac{1}{100} = .01.$$

For convenience in carrying forward our investigations, let us hereafter designate the probability of the event happening by the letter p, and the probability of its not happening by the letter q. By applying the rule already given, we find that the probability in the case stated of its not happening, is

$$q = \frac{990}{10 + 990} = \frac{990}{1000} = \frac{99}{100} = .99$$

If we add together the probabilities of its happening (.01), and of its not happening (.99), we have .01 + .99 = 1 for the sum. That is, expressed algebraically,

$$p + q = 1$$

Or, in words, the sum of the alternative probabilities that an event will happen and that it will not happen exhaust all the probabilities and make a certainty. The fractions are, then, complementary to one another. It follows that the value of either of them may be determined when the value of the other is known. For, by a simple algebraic process, we derive from the foregoing equations:

$$p = 1-q \text{ and}$$
$$q = 1-p$$

These simple deductions cover all the permutations of simple probabilities and they will be found very useful in the practice of actuarial science.

But probabilities are often complex, involving three or any number of events which may happen to the individuals of a group. Taken singly, the probability of each event may be computed by reference to the rule given; but the application of the same rules would hardly suffice to compute the value of the complex probabilities. For instance, the probability that both of two events would happen to a single individual would not be the sum of the probabilities that each would happen; for it is sufficiently plain that, unless one were the cause of the other, one might very well happen without the other happening. It is evident that the probability of all the events happening to one individual becomes smaller and smaller, the more the events are multiplied.

To illustrate this mathematically, let us take a group of 1,000,000, in which one event happens to 100,000, another to 10,000 and a third to 1000. Let us designate the probability of the first happening as p^1, the probability of the second as p^2 and of the third as p^3. By the application of the rule for simple probabilities, we find

$$p^1 = \frac{100,000}{1,000,000} = \frac{1}{10} = .1$$

$$p^2 = \frac{10,000}{1,000,000} = \frac{1}{100} = .01$$

$$p^3 = \frac{1,000}{1,000,000} = \frac{1}{1,000} = .001$$

Suppose the first event to have happened to its 100,000 individuals in the group; each individual in this new and smaller group of 100,000 to which the first event has happened, is on that account no more liable to the happening of the second event. We have found the probability of the second event (p^2) to be .01 for each individual, from which we find that in the smaller group of 100,000 only 1,000 would have the second event happen.

Taking this number (1000) as a new group, by applying the value of p^3 (.001), we find that of it only 1 person would have the third event happen. To but one out of the original 1,000,000, therefore, all three events would happen. The chance, then, of the three events happening to any one individual is not measured by adding the probabilities $(.1 + .01 + .001 = .111)$, but by multiplying them $(.1 \times .01 \times .001 = .00,000,1)$.

Expressed algebraically, this is $p^{1, 2 \text{ and } 3} = p^1 \times p^2 \times p^3$ or, in words, the probability that a number of events will happen to one individual is equal to the product of the probabilities that each will happen.

We have seen that the probability that an event will not happen is equal to unity less the probability that it will happen $(q = 1 - p)$. It is equally evident that the sum of the probabilities that all the events will happen to one individual and that at least one of the events will fail to happen exhausts all the probabilities and, therefore, also equals 1; from which it appears that the probability that at least one event will fail is equal to the difference between unity and the probability that all the events will happen.

Expressed algebraically, this is:

$$q^{1, 2 \text{ or } 3} = 1 - (p^1 \times p^2 \times p^3).$$

RUDIMENTS OF PROBABILITIES.—II.

We have considered the probability that all will happen and that at least one will fail to happen to one individual of a group and have found them to be complementary to each other. Let us now consider the probability that all will fail, which on the surface would have appeared to be complementary to the probability that all will happen. It is at once evident that the probability that all will fail is decreased instead of increased when the number of different events is multiplied.

Reverting to our group of 1,000,000, in which the first event will happen to 100,000, the second event to 10,000 and the third to 1000, we find that, taken severally, the probabilities that each will fail are as follows:

$$q^1 = \frac{900{,}000}{1{,}000{,}000} = \frac{9}{10} = .9$$

$$q^2 = \frac{990{,}000}{1{,}000{,}000} = \frac{99}{100} = .99$$

$$q^3 = \frac{999{,}000}{1{,}000{,}000} = \frac{999}{1{,}000} = .999$$

In the group of 1,000,000, the event fails to happen to 900,000; at the probability of the second event's failing to happen (q^2), it will fail to happen to 900,000 × .99 or 891,000 out of the group of 900,000 to which the first event has failed to happen. Again in this new group of 891,000, at the probability of the third event's failing to happen (q^3) it will fail to happen to 891,000 × .999 or 890,109.

We thus find the probability of the first event failing to happen to be $\frac{900{,}000}{1{,}000{,}000}$, of the first and second both failing to be $\frac{891,000}{1,000,000}$ and of all three failing $\frac{890,109}{1,000,000}$; of these fractions, the second is the product of the first by the value of q^2, the probability of the second failing (.99); and the third is the product of that product by q^3, the probability of the third failing (.999).

Algebraically expressed, this means

$$q^{1,2\text{ and }3} = q^1 \times q^2 \times q^3.$$

Just as the probability that all will fail is decreased by the multiplication of the different events, so the probability that at least one will happen is increased. And these two things are found to be complementary probabilities; for it is evident that to all in the original group of 1,000,000 to which all of the events have *not* failed to happen, at least one of them must have happened. Therefore the sum of those to which all the events failed to happen and of those to which at least one event happened equals the whole number. Consequently, the probability that all will fail plus the probability that at least one will happen is equal to certainty or unity; and the probability that at least one will happen is equal to unity less the probability that all will fail.

Algebraically expressed, this means

$$p^{1,2\text{ or }3} = 1 - (q^1 \times q^2 \times q^3).$$

To ascertain the probability that the first will happen and the other two events fail to happen, let us recur to the original group of 1,000,000, to 100,000 of which the first event happens, to 10,000 the second and to 1000, the third. Out of the 100,000 to which the first event happens, the second fails to happen to $q^2 \times 100,000$ or 99,000; and out of this 99,000, the third fails to happen to $q^3 \times 99,000$ or 98,901.

Thus we find the probability that the first will happen and the second fail is $\frac{99,000}{1,000,000}$ and that the third will also fail is $\frac{98,901}{1,000,000}$. The former of these fractions is equal to the product of p^1, the probability that the first will happen, into q^2, the probability that the second will fail ($.1 \times .99 = .099$); and the latter of these fractions is the product of this product into q^3, the probability that the third will fail ($.099 \times .099 = .09,890,1$).

Expressed algebraically, this means

$$p^1\, q^{2 \text{ and } 3} = p^1\, (q^2 \times q^3).$$

An equivalent result would be reached if we sought the value of the probability that the second would happen and the first and third fail; hence, it is evident that the formula is of general application.

We have found that the probability that two or more events will happen is the product of the several probabilities that each will happen ($p^1 \times p^2 \times \,..$); that the probability that two or more will fail is the product of the several probabilities that each will fail ($q \times q^2 \times \,..$); and that the probability that one will happen and the others fail is the product of the probability that one will happen into the probability that the others will fail ($p^1 \times q^2 \times q^3 \times \,..$). We might in the same way develop the fact that the probability that one will fail and the others happen is the product of the probability that one will fail into the probability that the others will happen ($q^1 \times p^2 \times p^3 \times \,..$). And it follows, in order, that the combined probability that two or more events will happen and two or more fail is the product of the probability that the two or more will happen into the probability that the two or more will fail.

Algebraically expressed, this is

$$p^{1,2}\, q^{3,4} = (p^{1} \times p^{2})\ (q^{3} \times q^{4}).$$

All other compound probabilities may be derived from an elaboration on these formulas.

Corollary 1. If events be such that in the nature of things they cannot concur, that is, occur to the same individual of the group, the probability that either will happen must be computed separately and the probability that one will occur is equal to the sum of the separate probabilities. The reason this does not apply to the probability that at least one will happen when the events can concur, is that in that case there is the additional probability that they may concur.

Corollary 2. If two events stand related as cause and effect or in such manner that the second cannot occur unless the first has happened, the situation is materially modified. In the latter case, the probability that the second event will happen is exactly the equivalent of the probability that both will happen, viz: $p^{1} \times p^{2}$.

SIMPLE PROBLEMS IN RATE-MAKING.

One year is the ordinary unit of time in insurance; premiums for a longer or shorter time are commonly deduced from the annual probability by some means, more or less scientific and exact. In practically all kinds of insurance, excepting only against death and illness, the hazard is tolerably uniform and, at least, does not necessarily increase continuously, resulting in certainty. Consequently, by separating out a group, to each of which, so far as *prima facie* appears, the event insured against is equally likely to happen, and observing to how many this event has actually happened in the space of one year, we can ascertain the probability of its happening to each of a like group to whom we furnish insurance.

Thus, for instance, suppose that we have separated out from statistical data a group of 1,000,000 dwelling houses which have been exposed to the risk of fire for one year each and that we find that of this number 1000 have burned completely, we find that the probability of a dwelling house being completely destroyed by fire within one year is $\frac{1000}{1,000,000} = .001$. This, however, evidently, does not cover the total probability of loss by fire, since it might easily happen that the damage done by fires which do not totally consume the buildings is yet greater. Therefore, let us transfer our attention from numbers to amounts, assuming the dwellings to be worth $1000 each, or an aggregate of $1,000,000,000 and the damage to be a total of $2,000,000. Applying our formula, we find the risk of damage by fire to be valued for each dwelling at $\frac{2,000,000}{1,000,000,000} = .002$. This would constitute the net premium, excepting that, if one wished to be very exact, he might ascertain at what period in the year, on the average, the losses occurred or were payable and discount the net probability for that term. This might be accomplished by reducing the total damage of $2,000,000 to a present value at the beginning of the year before dividing by the aggregate amount at risk to ascertain the probability. The quotient would be the net, discounted probability and constitute a premium, sufficient when improved at interest to meet the demands upon it as they accrue.

Whether numbers exposed and number of losses or amounts at risk and amount of losses should be employed in computing a probability for rate-making purposes, even when no partial damages enter into the account and all losses are total, has been a warmly disputed question. It is almost, if not quite, unavoidable to use amounts when making calculations of probabilities, involving partial damage. Where this is not the case, the result would be the same, whether by number or amounts, if the amount of each individual risk were the same. When the amounts of the individual risks are different, it appears clear that the premiums should vary in direct ratio to the amount at risk; which would indicate that the probability of loss should be ascertained for some unit of amount at risk and this net premium be increased or diminished according as the insurance is for a greater or less amount than this unit. But the probability deduced from numbers will give the basis for the probability of loss for any unit of amount at risk; and it follows that the correct system, when dealing with cases not involving partial damages, is to compute the probability from the numbers involved. The distinction is, however, ordinarily of no great moment.

The assumption, in selecting one year as the unit of time, that the risk of loss, one year with another, is a reasonably uniform one, is scarcely true in more than a few branches of insurance, possibly such as individual accident. In fire, there is the conflagration hazard which actuaries have never had a fair opportunity to compute; these conflagrations recur in periods commonly much longer than one year, caused by immediate exposures as in cities or by wood or prairie fires. In credit, surety and allied lines of insurance there is the panic hazard, also recurring commonly at longer periods than one year. If an opportunity were offered actuaries to devise from the data, which insurance companies might furnish, premiums and methods of reserve to meet these extraordinary contingencies, the computations would doubtless involve operations as nice and as difficult as any arising in life insurance to the problems of which actuarial attention has been mainly directed.

In making premiums for a shorter term than one year, commonly more than a mere *pro rata* part of the annual premium is demanded. This addition to the annual rate is made arbitrarily and more as a penalty than to cover any increase in the hazard. In fact, it is generally assumed in actuarial calculations that the hazard is uniform throughout the year. As will be seen later in these studies, actuaries do sometimes treat the risk of death as a regularly increasing one throughout each year; but most of their calculations proceed on the contrary assumption, which is, in fact, quite as near the fact. The fact is that the law of mortality, while obedient to a general tendency to increase, after the era of childhood is past, is not uniform throughout short periods of time. Mortality at all ages is increased at certain seasons of the year by variable weather, excessive heat or cold or humidity. Likewise, fires are more common and destructive in the coldest weather; and there are two or more seasons of liquidation of commercial credits every year when the risk of loss in credit insurance is augmented. Notwithstanding which, no attempt has been made to vary the rates in accordance with these fluctuations in the hazard. In fire insurance, the arbitrary method which has been adopted yields rates which are, perhaps, always high enough. The rule is: charge .2 the annual rate for one month and .1 additional for each month up to and including six, making .7 for six months; and .05 additional for each month beyond six, making .95 for eleven months.

The system for computing premiums for longer than one year in fire insurance, known as "term premiums," is equally arbitrary but does not sin on the safe side. The common rule is to charge twice the annual premium for three years, two and one-half times for four years and three times for five years. The absurdity of so great a reduction has been so far recognized that on certain classes of risks the companies now charge two and one-half times the annual premium for three years, three times for four and three and one-half for five. No effort has been made to solve the problem on scientific grounds.

To do so offers no serious difficulties. Given an aggregate insurance of $1,000,000,000 and an aggregate annual loss of $2,000,000 or .002, probability of loss or net premium. The losses on the $1,000,000,000 insured will, for five years, run as follows:

First year.	Amount at risk	$1,000,000,000
	Loss @ .002 thereon	2,000,000
Second year.	Amount at risk	998,000,000
	Loss @ .002 thereon	1,996,000
Third year.	Amount at risk	996,004,000
	Loss @ .002 thereon	1,992,008

Fourth year.	Amount at risk.	$994,011,992
	Loss @ .002 thereon.	1,988,024
Fifth year.	Amount at risk.	992,023,968
	Loss @ .002 thereon.	1,984,048

Let us assume that the losses of each year are payable at the beginning of that year (annual premiums are thus payable), and then proceed to find what would need to be on hand at the beginning of the first year to pay the first year's losses, the second year's, etc. Assuming that the money will earn interest at five per cent per annum until needed, we discount at that rate with the following results:

Sum at beginning of first year to meet losses of first year. . .	$2,000,000
" " " " second " . .	1,900,952
" " " " third " . . .	1,806,809
" " " " fourth " . .	1,717,331
" " " " fifth " . . .	1,632,280

Having reduced the data to this form, to find the amount we need to have on hand to pay the losses of the first and second years, the first three years, etc., we merely add these sums for the respective years included, thus:

Sum beginning of 1st year to meet losses of 1st and 2nd years.	$3,900,952
" " " " first three "	5,707,761
" " " " " four "	7,425,092
" " " " " five "	9,057,372

To ascertain the premium, that is the part which each dollar of insurance would need to contribute, divide the whole amount needed by the sum originally insured, $1,000,000,000, which gives the following results:

Advance premium for two years.	.0039 +
" " three "	.0057 +
" " four "	.007425
" " five "	.009057

In reaching this result, we have obtained the loss for the second year by the product of p, the probability of loss into the insured property surviving the year; and so with the loss of each year. We have then discounted them to present value and added the discounted values, dividing the sum by the original sum insured. Precisely the same results may be obtained by an analogous process, dealing with the probabilities only, thus: to obtain the part of the combined advance premium covering the hazard of the second year, multiply the probability of surviving the first year (q^1) by the probability of loss the second year (p) and multiplying the product by the present values of \$1, discounted one year at five per cent (v^1); etc., throughout the series of years. The sum of these parts of the advance premium make up the whole and the formula may be stated generally, thus:

Net advance prem. $= p + q^1 pv^1 + q^2 pv^2 + q^3 pv^3 + \dots$

This simple process and formula will hereafter be found to contain the principle which solves many intricate problems of life insurance rate-making.

PROBABILITIES OF MORTALITY.

We have seen that to ascertain the probability of an event, it is first requisite to separate out a group, to each of whom the event is *prima facie* equally likely to happen. Then, by observing this group for any certain space of time, it separates into those to whom the event has not happened and those to whom it has happened. The probability of the event happening to each of the group is then calculated by dividing the number to whom it has happened by the number of the original group.

Suppose, now, that, instead of a group to each of which the event was *prima facie* equally likely to happen, a heterogeneous group were taken, to the individuals of which the event was *prima facie* likely to happen in widely varying degree. Suppose, for instance, that the group contained dwellings, warehouses, saw-mills and powder magazines and it were attempted to calculate the probability of loss by fire to any one of these by dividing the aggregate loss by the aggregate values exposed. It is at once clear that the quotient would not represent accurately the probability of loss to any one of the classes of property; but would, instead, be less than the probability of loss to the more hazardous and more than the probability of loss to the less hazardous. No fire insurance company would proceed thus to apply the same rate of premium to unequal hazards, and, if it did so, experience would soon demonstrate the error; for the owners of property, by a process of conscious selection, would insure more hazardous property and leave the less hazardous uninsured.

It is *prima facie* clear that the probability of decease is different at different ages, and that after the period of childhood is passed, the probability of decease increases with age with a certain regularity. In youth and up to middle age, the increase may be slow and might be disregarded; but there is a very considerable difference between the probability of death at age twenty and the probability at age fifty, other things being equal. It follows that a fraction formed by dividing the number of deaths in a group of persons at different ages by the number of the group would not represent the probability of death of persons at any one of those ages. It would also not represent the aggregate probability of death in any other heterogeneous group, unless the group were formed of a proportionate number of persons at each age.

Notwithstanding which, there has been a practice in certain life insurance organizations in the United States to charge the same for a year's insurance without regard to age; this was brought about by raising money to pay death-losses by means of an equal levy upon all the insured, without regard to age. The effect was to make premiums as if the probability of decease were the same at all ages and, also, as if the probability of an individual's death were the same in any one year as in any other year.

The first of these assumptions has come to be universally conceded to be preposterous and the practice of level charges at all ages has accordingly been generally abandoned. The absurdity of the second assumption is quite as evident, but it is not yet so universally recognized. In consequence, we find some life insurance organizations violating all the principles of probabilities by proceeding on the basis that the probability of decease remains throughout life identical with the probability of decease at the age of entry into the organization. This is *prima facie* erroneous, and the consequences can hardly fail to be mischievous.

If we started with a group from a definite point of time, to which none were ever added and which was diminished only by the deaths, the consequence of this system would merely be to apportion the losses by fixed ratios which did not correspond with the actual hazards after the first year, provided the premiums were not fixed. If it were attempted to furnish insurance for premiums fixed according to the probabilities of death at age of entry, the consequence would be that the premiums of the second year would not cover the aggregate loss, since the probability of death has increased for each insured person, while his premium has remained the same—namely, just sufficient to cover the probability of death the year before. And this deficiency would increase every year. The premiums being fixed, the admission of a fresh group each year would not help matters, as the new group would, according to the premises, pay in only enough to cover their own probabilities. The premiums not being fixed on the admission of a fresh group each year, the apportionment would place an unfair proportion of the loss-burden on the new group and this would become heavier with each year.

From these considerations, it is apparent that if we are to determine accurately the probabilities of decease, for any individual at any age, we must do so by discovering how many persons die out of a group of persons at that age and otherwise with *prima facie* equal probabilities. It is also apparent that the probability of the death of the same individual when he attains the next higher age must be independently ascertained from the mortality in a group of persons at that age. And so on throughout all ages. Insured lives, though at the outset carefully selected, are not, on the whole, select or healthy lives, because of the circumstance that commonly the renewal of a policy is not optional with companies. Consequently, the probability of decease at a certain age, in order to be useful in insurance, should be derived from average lives, such as obtain in a company in which the benefit of fresh selection has somewhat worn off. Thus this probability is not that of the death within the year of a man who begins it in good health, but the probability of a man in average condition as to health.

ANNUAL LIFE CONTINGENCIES.

We have seen that the proper way to determine the probability of decease or survival is by observing the number dying during a certain period out of a number setting out at the same age and with apparently similar chances of survival. This certain period is naturally measured by the unit of one year and the probability of survival or decease in a period longer than one year may be divided into a succession of probabilities during one year, followed by a fraction or not, according as the whole period is an exact number of years or not. For convenience, therefore, actuaries make the unit of measurement the year from birthday to birthday.

If a large number of persons could be put under observation who were born on the same day, by following them down through the diminution of their number by death until all were deceased, we could determine, one after another, the probabilities of death and survival for each year of age. For the number starting out from each age, the number dying before attaining the next age and the number surviving to that next age would be ascertained for each age of life; and from these three data all the probabilities of death and survival could be calculated.

For instance, let l_0 be the number born in the same year and d_0 the number dying in that year, l_1 the number surviving the first year and setting out upon the second, d_1 the number dying during the second, etc.; then, first, it is apparent that

$$l_1 = l_0 - d_0, \text{ or, generally,}$$

$$l_{x+1} = l_x - d_x$$

Let us designate the probability of survival as p; then by applying the rule for determining the value of a simple probability, we find that the probability of surviving the first year of life is measured by the fraction found by dividing the number of the survivors by the number who set out, thus:

$$p_0 = \frac{l_1}{l_0} = \frac{l_0 - d_0}{l_0} = 1 - \frac{d_0}{l_0}, \text{ or, generally,}$$

$$p_x = \frac{l_{x+1}}{l_x} = \frac{l_x - d_x}{l_x} = 1 - \frac{d_x}{l_x}$$

Similarly, designating q for the probability of not surviving, that is, of dying, during the year, we have by applying the same rule, a value for the probability of decease measured by the fraction found by dividing the number who died by the number who set out, thus:

$$q_0 = \frac{d_0}{l_0} = \frac{l_0 - l_1}{l_0} = 1 - \frac{l_1}{l_0}, \text{ or, generally,}$$

$$q_x = \frac{d_x}{l_x} = \frac{l_x - l_{x+1}}{l_x} = 1 - \frac{l_{x+1}}{l_x}$$

It would, however, be very difficult, if not impracticable, to collect such a group and keep them all under observation until the last one died; and it is by no means necessary. For, since we are dealing with the probability of death during years of age, it is evidently a matter of no importance to also have concurrence as to calendar years. It is enough if a large number have been observed from birth to decease, the years of age when the respective deaths occurred being marked, so that it is possible to group the different years of life together. And, by this method, the probabilities for one year of age may be determined absolutely independently of any other year, if some large number of persons entering upon that year of age has been observed for that year of age and the deaths noted. It is, in fact, only by computing these probabilities separately that anything can be accomplished in actual practice, since the lives under observation are constantly shifting and it is impossible to keep track of any large group throughout a long term of years.

This is true concerning insured lives, the insured coming in and going out of the company at all sorts of intervals. Consequently, probabilities which are computed from deaths and survivals among insured lives are separately and individually determined for each year of age. To do this, all the individuals who entered upon that year of age during the continuance of their insurance are collated, the number who died during that year of age is computed, and from these data the probabilities are readily calculated by the use of formulas already given.

But a difficulty is at once met when it is undertaken to collate the number entering upon any given age, namely, that the insured entered at all fractions of the year of age and that, in consequence, it would be very difficult to group them by exact years of age. This difficulty is, however, practically obviated by the custom of the companies to reckon ages from the half-way point between birthdays, which assures that the persons insured as of a certain age are on an average nearly exactly of that age, about as many having been received who were younger as who were older.

Life contingencies are also computed from census statistics; but, because no such precaution is taken in collating such statistics as in receiving applicants for insurance, it happens that all the persons who have not passed their next birthday are ranked as of their age at the last birthday. Consequently they are, on the average, one-half year older than when at the last birthday. The deaths, on the contrary, are given just as we would desire them, namely, covering all who die while in a certain year of age. On the assumption, then, that the deaths are reasonably uniform throughout the year, it appears that the number surviving to one-half year beyond the exact age is smaller than the number setting out at that age by just one-half the number who have died.

MORTALITY TABLES.

We have seen in our last lesson that if we could keep under observation a large number of persons who were born on the same day, until all have died, we would be able to compute successively the probabilities of death and survival for each year from birth to the extreme limit of life. We shall now see that in the course of the observations we should also form a complete table of survivors and decedents for each age, which we call a mortality table.

Starting from birth, we obtain successively the number who survive the first year of life, the second year of life, etc., up to the extreme limit of life. The differences between these each year show the respective decedents. Special marks which distinguish this table from a mere collection of data from which the probabilities are independently deduced for each age, are that the radix or number setting out from any age less the number of decedents during that year of age gives not merely the survivors of that group, but also the radix or the group setting out from the next age. Thus the data become connected and associated in such manner that the table will show not merely the survivors and decedents for one year but, at a glance, for any integral number of years, out of a group starting from any age. This tabulated form will be found very useful in computing complex probabilities.

The practical impossibility of making such a table in this manner has already been pointed out and is apparent. Not only was information such as this not attainable when the earlier tables were made, but information such as we can now obtain from censuses or from the returns as to insured lives was also not to be had. For much more than a century after an attempt was made to develop the law of mortality, the only data which could be obtained were transcripts of the deaths from parish registers. The first mortality table was constructed by Graunt from London registers and the second by the astronomer Halley from the registers of Breslau, Germany. The registers furnished no information as to the number who were exposed at any age. The manner of constructing the mortality tables was to assume that the data fairly represented the mortality which would successively apply to a group equal to the whole number of decedents, born on the same day.

For instance, in the case of the table constructed from the statistics of Breslau, the total number of deaths was 1174, which number was accordingly taken as the radix of the table. The deaths before attaining the age of one year were 348, from which the rate of mortality was made $\frac{348}{1174}$, or 29.64 per cent, and the number of survivors $1174 - 348 = 826$. The fact was, however, that these deaths at less than one year of age were among 1238 children, born within the period covered by the observations of mortality instead of among 1174. And it must be evident that, unless the population were precisely stationary, the births balancing the deaths each year and nobody removing from or into the field of observation, the assumption that the mere number of deaths during any period could furnish a radix for a mortality table constructed as if from lives observed from the time of birth was sure to be erroneous. As the populations of Breslau and other places, which furnished statistics used to compile these tables, were in each case increasing, it followed that the radix was in each case too small to correctly represent the mortality at the younger ages and this, in turn, made the mortality proportionately too small at older ages. Where these tables were used for insurances, this proved an error on the safe side; where used for annuities, it was an unfortunate error.

The observation of insured lives and the lives of annuitants indicated the faults of these tables, and, parenthetically, it may be said, that they were found faulty in just the respects which have been mentioned and which might have been suspected from the fault of the system.

The making of tables from data which independently demonstrate the probabilities of decease and survivorship for each age, is attended by somewhat greater difficulties than the mere employment of the whole number of deaths as a radix, as in the system just described. For instance, statistics from the census or from observations on insured lives, when reduced to simple forms, set forth the decedents and survivors out of a certain number of years of life exposed, all the figures being different at the various ages. It is not possible to trace directly from these data the annual diminution of a group of actual individuals by death up to the extreme limit of life. In fact, the individuals are wholly lost sight of when they pass from one group into another; they merely appear as integers of years of life exposed.

What can be done is this: we may compute from these data the probability of survival and decease for each year of age independently and we will find it possible to construct a mortality table by using these ratios. For, starting out from any large group, such as 1,000,000, as a radix, we may, by applying the values of p_x and q_x (the probabilities of survival and decease), successively ascertain the number of decedents and survivors, using the survivors of one year as the radix for computing the decedents and survivors of the next. Thus we may as well express the law of mortality as if it were possible to put under observation such a number born on the same day until exhausted by death.

To do this from data taken from the census, we may determine the probabilities of death and survival from what is known as the meridian. mean or central death rate, which means the number dying during any year of age divided by the number surviving to one-half year beyond the initial age. This central death rate is designated by the algebraic expression, m_x. We may trace its relations to p_x and q_x as follows: By our former lessons we have found that $p_x = \frac{l_{x+1}}{l_x}$ and $q_x = \frac{d_x}{l_x}$.

We now find that $m_x = \frac{d_x}{l_x - \frac{1}{2}d_x} = \frac{l_x - l_{x+1}}{l_x - \frac{1}{2}d_x}$

We may derive a more workable formula, however, by the following ingenious process. Suppose

$$p_x = \frac{900}{1000} = \frac{1000 - 50 - 50}{1000 - 50 + 50} = \frac{950 - 50}{950 + 50} = \frac{1 - \frac{50}{950}}{1 + \frac{50}{950}}$$

$$q_x = \frac{100}{1000} \text{ and } m_x = \frac{100}{950}.$$

Expressing $\frac{50}{950}$ in terms of m_x, we have

$$p_x = \frac{1 - \frac{1}{2} m_x}{1 + \frac{1}{2} m_x} = \frac{2 - m_x}{2 + m_x}$$

and $q_x = 1 - p_x = 1 - \frac{2 - m_x}{2 + m_x} = \frac{2 + m_x - 2 + m_x}{2 + m_x}$

$= \frac{2\,m_x}{2 + m_x}$, which values may then be used directly in constructing the mortality table.

MORTALITY TABLES.—GRAPHIC SYSTEMS OF GRADUATION, I.

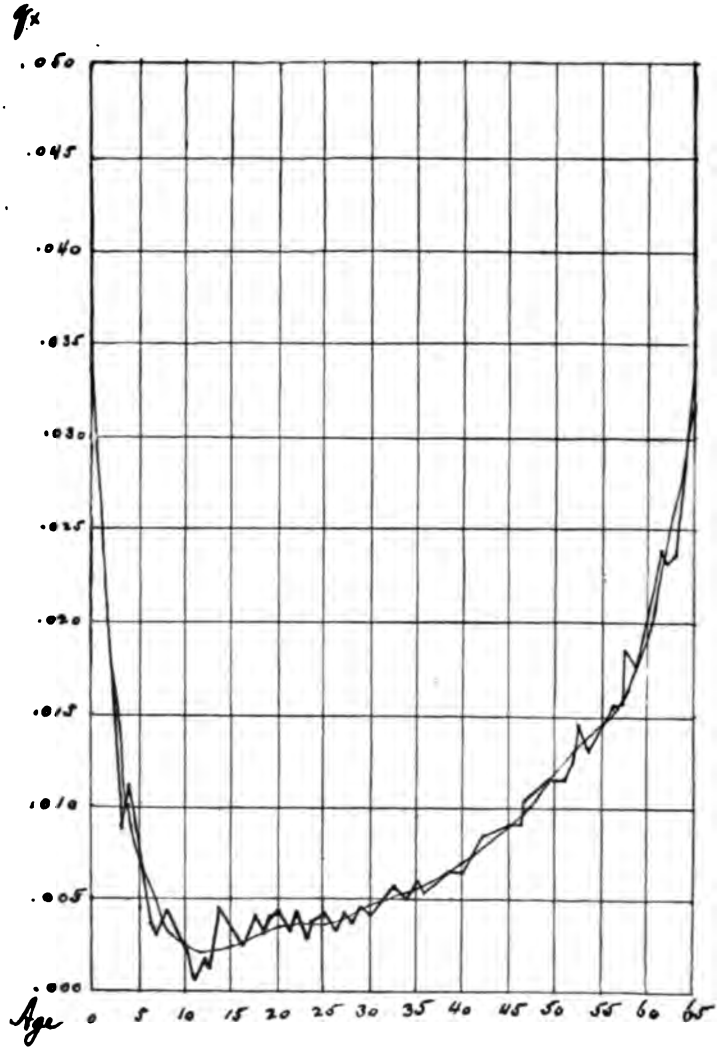
UNLESS the number of lives dealt with is at each age very large, the values of the probabilities of death and survival derived from them will not be as uniform as is desirable. Some things, for instance, are certainly known about the rate of mortality. Both from what is *prima facie* reasonable and from the testimony of a large aggregation of statistics, we know that the rate of mortality tends to fall from infancy up to about ten years of age and then gradually to increase. The increment itself is also at an augmenting ratio, so that in old age the increase is very rapid. The increase during the years of men's prime, say from twenty to fifty, is not so much the direct result of aging as it is of the fact that merely living longer multiplies the chance that the system is already undermined by disease. In other words, the probability of survival for one year may not be much different for either of two lives, freshly selected by medical examination, though one be at twenty years of age and the other at fifty. But the expectation or probable life of the former will be much longer than that of the latter, and the hazard of the former dying during his fiftieth year, if he survives so long, is much greater than of his dying during his twentieth year, for there is the hazard of his not reaching fifty in unimpaired health.

This last hazard applies to each of the group, forming the radix of a mortality table. Consequently, independently of the increase of mortality from actual old age, we expect an increase with each year (after childhood has been safely passed) because of deteriorated average health conditions. Deviations, such as a decrease of mortality, will be found to be followed at a greater or less interval by a sudden and disproportionate increase, both phenomena being the results of the inadequate number of lives dealt with. In most cases, therefore, it is to be expected that a mortality table, constructed as per the instructions given in the last lesson, will exhibit glaring irregularities and want of uniformity.

As has already been indicated, these irregularities counterbalance each other and the correct ratios may, therefore, be arrived at by some system of graduation which will spread out the excesses so as to fill up the deficiencies and yield a uniform table, showing a gradual but regular increase in the mortality.

No method has yet been invented to accomplish this which is so simple and so self-explanatory as the graphic method. The principle is to lay out the trend of mortality, as developed by the actual investigation, upon a scale, drawing lines so as to connect the values of q_x at the different ages, thus representing the force of mortality by a broken line, as in the accompanying illustration. This exhibit is taken from the mortality experience of the Australian Mutual Provident Society and excellently illustrates the graphic method of graduation.

The vertical spaces represent a scale of mortality differing by .005 per unit of the number of lives exposed. The variations in the force of the mortality as expressed by the ratios deduced from the society's actual experience are indicated graphically by the broken line which starts at nearly .034 and rapidly descends until at age three it is less than .009, and then after several reactions it is at about .001 at age eleven Then it reascends, at first very slowly and with many variations up and down


q_x
.050
.045
.040
.035
.030
.025
.020
.015
.010
.005
.000
Age
0
5
10
15
20
25
30
35
40
45
50
55
60
65

until age thirty, after which time the upward tendency very strongly asserts itself.

It must be at once clear that the trend of this broken line may be closely approximated by a more or less regular curve, which will avoid the violent variations, while at the same time representing on the average faithfully the facts which the unadjusted table presents. This curved line must also correspond with the known characteristics of the rate of mortality enumerated in the beginning of this lesson. Thus prepared for the task, we may sketch freehand, but with care, the curved line in the diagram, keeping a middle course. From this curved line, by reference to the scale, the ratios of mortality for the different ages may be taken and a graduated mortality table constructed by successive application of these ratios, in accordance with the instructions in the last lesson.

This diagram is reduced to small proportions, so as to print in the page of an ordinary book. In actual practice, there are advantages in occupying more space and using a larger scale, and having paper subdivided accurately into very small intervals.

MORTALITY TABLES. — GRAPHIC SYSTEM OF GRADUATION, II.

Another system of graduation, which calls for more space for its successful development, is to construct a similar curve from the data of an ungraduated mortality table. The unadjusted table is first constructed by taking the values of q and p (the probabilities of death and survival for each year of age), direct from the original data and proceeding as per instructions on pages 62–3 of this book to construct a table, taking any large number of lives as a radix. A good specimen of an unadjusted table is found in the published tables of the experience of the Canada Life Assurance Company and runs as follows:

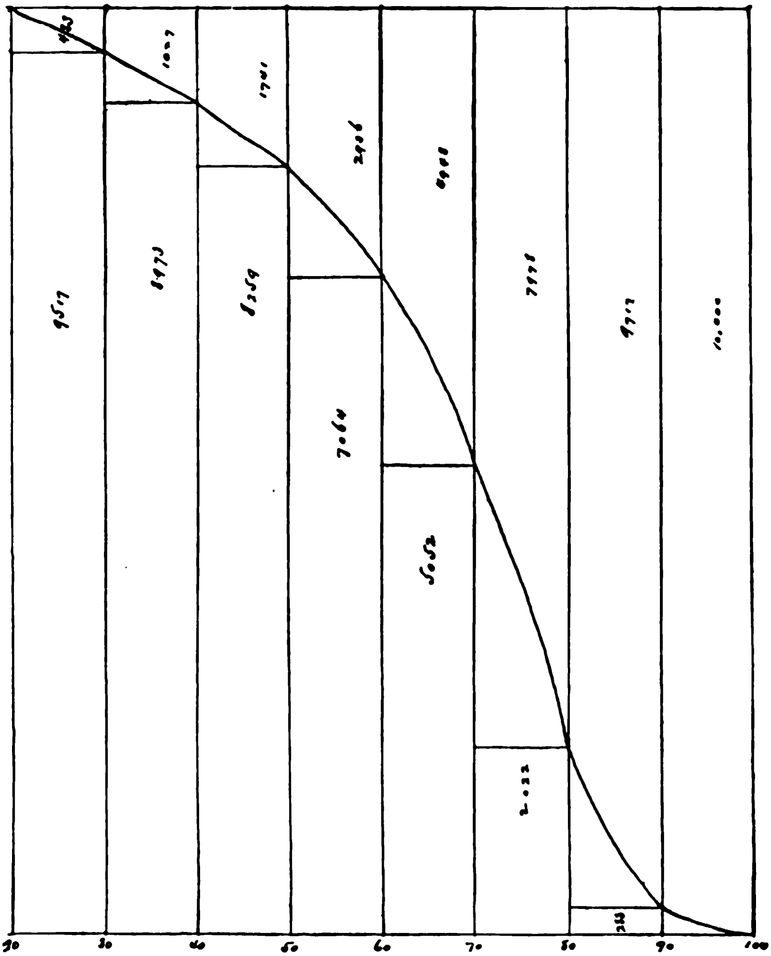
AGE.	l_x	d_x	AGE.	l_x	d_x	AGE.	l_x	d_x
20.......	10,000	53	38.......	9,090	54	56	7,631	119
21.......	9,947	33	39.......	9,036	63	57.......	7,512	126
22.......	9,914	55	40.......	8,973	71	58.......	7,386	146
23.......	9,859	44	41.......	8,902	58	59	7,240	176
24.......	9,815	52	42.......	8,844	53	60.......	7,064	177
25.......	9,763	58	43.......	8,791	65	61.......	6,887	174
26.......	9,705	54	44.......	8,726	67	62.......	6,713	198
27.......	9,651	43	45.......	8,659	72	63.......	6,515	194
28.......	9,608	43	46.......	8,587	78	64.......	6,321	168
29.......	9,565	48	47.......	8,509	88	65.......	6,153	204
30.......	9,517	51	48.......	8,421	83	66.......	5,949	217
31.......	9,466	45	49.......	8,338	79	67.......	5,732	220
32.......	9,421	45	50.......	8,259	89	68.......	5,512	229
33.......	9,376	54	51	8,170	94	69.......	5,283	231
34.......	9,322	61	52.......	8,076	91	70.......	5,052	221
35.......	9,261	59	53	7,985	113	71.......	4,831	276
36.......	9,202	61	54	7,872	127	72.	4,555	333
37.	9,141	51	55	7,745	114	73.......	4,222	274

AGE.	$_x$	d_x	AGE.	l_x	d_x	AGE.	l_x	d_x
74	3,948	293	81	1,694	266	88	398	27
75	3,655	291	82	1,428	278	89	371	88
76	3,364	256	83	1,150	249	90	283	106
77	3,108	333	84	901	142	91	177	71
78	2,775	410	85	759	96	92	106	...
79	2,365	343	86	663	159	93	106	...
80	2,022	328	87	504	106	94	106	...
						95	106	...
						96	106	35
						97	71	71

It will be observed that the deaths and survivals under this table for ten year periods are as follows:

Inclusive.	Radix.	Deaths.	Survivors.
20–29	10,000	483	9,517
30–39	9,517	544	8,973
40–49	8,973	714	8,259
50–59	8,259	1,195	7,064
60–69	7,064	2,012	5,052
70–79	5,052	3,030	2,022
80–89	2,022	1,739	283
90–99	283	283	.....

Lay out a line, divided into eight equal spaces to correspond with the eight terms of ten years each. From the left hand point of the line, construct a perpendicular of an arbitrary length to represent the radix at age twenty. Complete the parallelogram by drawing the necessary lines. From each point on the base line, 30, 40, 50, 60, 70, 80 and 90, draw lines parallel with the perpendicular and ending in the top line of the parallelogram. On the first perpen-


483
9517
1007
8993
1741
8259
2906
7094
4948
5052
7778
2222
9717
283
10,000
20
30
40
50
60
70
80
90
100

dicular, beginning at the base, lay out a line proportional to the whole perpendicular as the survivors of 20-29 to the radix of 10,000 and draw a line parallel with the base connecting this point with the second perpendicular. This last line divides the parallelogram contained between the two perpendiculars into two parallelograms, which are proportional, the upper to the deaths for the period and the lower to the survivors of the period. Repeat this operation for each of the perpendiculars. Then, describe a curve, touching each of these outer angles, beginning at the upper extremity of the first perpendicular and ending at the right extremity of the base line, as in the diagram given herewith.

It is possible to derive a similar curve in a somewhat similar manner from population statistics, and also even without first constructing an unadjusted mortality table. In this case, as was pointed out in an earlier lesson, the number living in each year of age is, approximately, the number who attain one-half year beyond that exact age and so it is neither the radix nor the number of survivors. This brings it about that the principal parallelograms are divided by lines somewhere between the lines drawn parallel to the base in this diagram and the top line. And, accordingly, the curve should be described, starting from the first perpendicular, so as to contain below itself as much space as was contained by the original parallelograms, the spaces cut off and the spaces added equalizing one another. To describe such a curve, it is not even necessary for the data to be rearranged so as to conform with a radix; all that is necessary is that each principal parallelogram be divided into two parallelograms, proportional to the number living and number dying during the age terms, 20-29, etc. This method was first described by J. Milne, author of the Carlisle Tables, and was by him employed in the construction of those tables.

The values of the probability of survival, p_x, and of the probability of dying, q_x, may be taken from these tables by use of dividers and a table of proportional parts. If these values are not quite so regular as may be desired—usually owing to drawing the diagram on too small a scale—they may be again graduated by reference to the first method of deriving those values by the graphic system, described in the last lesson. Sufficient accuracy may be attained by this system to determine the l_x and d_x columns at ages up to 65 or 70; but the scale used in the diagram would need to be very large to give satisfactory figures for the highest ages. The same difficulty was encountered in the system last described and is encountered in most of the systems in use.

MORTALITY TABLES.—FINLAISON'S METHOD OF GRADUATION.

The following method was introduced by Mr. John Finlaison and was said by him to have been used in the graduation of the Government Annuity Tables of 1829. If so, however, the results must have been further adjusted by some other method, because the table cannot be exactly reproduced by following this formula.

Let p''_x = the probability of survival of one year at age x, according to the original unadjusted table.

Let $p'_x = \frac{p''_{x-2} + p''_{x-1} + p''_x + p''_{x+1} + p''_{x+2}}{5}$; that is, equal one-fifth precisely of the five unadjusted values of p', with p'_x as the central value.

Then let $p_x = \frac{p'_{x-2} + p'_{x-1} + p'_x + p'_{x+1} + p'_{x+2}}{5}$; that is, equal one-fifth precisely, of the five unadjusted values of p', with p'_x as the central value.

The formula, $p_x = \frac{p'_{x-2} + p'_{x-1} + p'_x + p'_{x-1} + p'_{x+2}}{5}$ may be expanded, as follows, by substituting the values of p'' for p':

$$\begin{aligned} p_x = \left(\tfrac{1}{5}\right)^2 \{ & (p''_{x-4}+p''_{x-3}+p''_{x-2}+p''_{x-1}+p''_x) \\ & +(p''_{x-3}+p''_{x-2}+p''_{x-1}+p''_x+p''_{x+1}) \\ & +(p''_{x-2}+p''_{x-1}+p''_x+p''_{x+1}+p''_{x+2}) \\ & +(p''_{x-1}+p''_x+p''_{x+1}+p''_{x+2}+p''_{x+3}) \\ & +(p''_x+p''_{x+1}+p''_{x+2}+p''_{x+3}+p''_{x+4}) \} \end{aligned}$$

$$= \frac{p''_{x-4}+2\,p''_{x-3}+3\,p''_{x-2}+4\,p''_{x-1}+5\,p''_x+4\,p''_{x+1}+3\,p''_{x+2}+2\,p''_{x+3}+p''_{x+4}}{25}$$

$$=\frac{5\,p''_x+4(p''_{x-1}+p''_{x+1})+3(p''_{x-2}+p''_{x+2})+2(p''_{x-3}+p''_{x+3})+(p''_{x-4}+p''_{x+4})}{25}$$

$$=.20\,p''_x+.16(p''_{x-1}+p''_{x+1})+.12(p''_{x-2}+p''_{x+2})+.08(p''_{x-3}+p''_{x+3})+.04(p''_{x-4}+p_{x+4})$$

The graduated value of p_x by this method, is influenced therefore by the unadjusted values, p'', at the four preceding and the four succeeding ages.

The actual work may be done in strict accordance with the final formula, in which case the preliminary work should be the summation of $p''_{x-1} + p''_{x+1}$, $p''_{x-2}+ p''_{x+2}$, $p''_{x-3} + p''_{x+3}$ and $p''_{x-4} + p''_{x+4}$ for all values of x; the next, their suitable modification by the percentages; and the last, the assembling.

The work may also be done by differences as follows:

$$p_{x+1}=\tfrac{1}{25}\{5\,p''_{x+1}+4(p''_x+p''_{x+2})+3(p''_{x-1}+p''_{x+3})+2(p''_{x-2}+p''_{x+4})+(p''_{x-3}+p''_{x+5})\}$$

$$p_x=\tfrac{1}{25}\{5\,p''_x+4(p''_{x-1}+p''_{x+1})+3(p''_{x-2}+p''_{x+2})+2(p''_{x-3}+p''_{x+3})+(p''_{x-4}+p''_{x+4})\}$$

$$\Delta\,p_x=\tfrac{1}{25}\{p''_{x+1}+p''_{x+2}+p''_{x+3}+p''_{x+4}+p''_{x+5}-(p''_x+p''_{x-1}+p''_{x-2}+p''_{x-3}+p''_{x-4})\}$$

$$\Delta\,p_{x+1}=\tfrac{1}{25}\{p''_{x+2}+p''_{x+3}+p''_{x+4}+p''_{x+5}+p_{x+6}-(p''_{x+1}+p''_x+p''_{x-1}+p''_{x-2}+p''_{x-3})\}$$

$$\Delta^2 p_x=\tfrac{1}{25}(p''_{x+6}+p''_{x+4}-2\,p''_{x+1})$$

Calculate p_x and p_{x+1} for an initial age (x) by the other formula and from these $\Delta\,p_x = p_{x+1}- p_x$; then calculate $\Delta^2 p_x$ for every value of x by the foregoing formula; then get the values of $\Delta\,p_x$ from the formula; $\Delta\,p_{x+1} = \Delta\,p_x + \Delta^2\,p_x$; and, finally, get the values of p_{x+1} by repeated additions of $\Delta\,p_x$, thus:

$$p_{x+1} = p_x + \Delta\,p_x$$

$$p_{x+2} = p_{x+1} + \Delta\,p_{x+1}$$

$$p_{x+3} = p_{x+2} + \Delta\,p_{x+2}, \text{ etc.}$$

The simplest way, however, to compute the value of p_x by this method, is as follows: Put the unadjusted values of p'' in a column; add the five values of which p''_x is the central value and put the sum in a second column opposite age x; add the five values in this column, in turn, having $5\ p'_x$ as the central value and put the sum in a third column opposite age x; divide by 25 to get p_x and put in a fourth column opposite age x. The following shows the work.

AGE.	p''_x	$5\ p'_x$	$25\ p_x$	p_x
25........	.99486			
26.........	.99308			
27.	.99353	4.96628		
28........	.99217	4.96316		
29.........	.99264	4.96272	24.81342	.99254
30.........	.99174	4.96087		
31.........	,99264	4.96039		
32........	.99168			
33........	.99169			

This method could produce the true value of p accurately, only in case the values formed an arithmetical progression. It is correct to first differences only.

The first value thus obtained is for the fifth age beyond the age at which the original data began; in the illustration given, age 29, when the data began at age 25. The four values, p_{28}, p_{27}, p_{26} and p_{25}, will usually be found very easily by differencing. The same difficulty will, however, be found at the other end of the column; and often, likewise, the data is too scanty at the higher ages to give anything like reliable results. Resort is frequently had, therefore, to Gompertz's formula, described in a later chapter, to fill out the table.

MORTALITY TABLES—GRADUATION BY FORMULA.

We have already discovered in our inquiries into methods of graduation of mortality tables that the curve, whether it be the curve in a tabulation of the rate of mortality (q_x) or the central death-rate (m_x) or of the l_x column in the completed mortality table, is governed by a law which may be traced out by the differences. This gives us the clue that there is a law of mortality which may well govern the graduation of all tables. The application of this law will, of course, vary according to the special data, drawn from experience, upon which each table may be based.

The first attempt to formulate such a law was by De Moivre in his "Treatise of Annuities on Lives," published in 1725, before any considerable quantity of mortality statistics was available. His theory was that while the rate of mortality increased for each year of life, an arithmetical series with equal decrements for each year would make up a mortality table. Thus, assuming seventy-four people to be living at age twelve, his theory was that one might be expected to die each year, terminating the series at age eighty-six. This means that the d_x column in a mortality table should remain constant, the same number. Thus in order to graduate a mortality table no information would be requisite other than the radix number of lives and the extreme age attained; the constant decrement would then be obtained by dividing the radix number by the difference between the radix and the extreme ages, thus assuming the radix age to be ten, the extreme age 100 and the radix number 100,000.

$$d_x = \frac{100{,}000}{100-10} = \frac{100{,}000}{90} = 1{,}111+$$

And from this a mortality table would be formed by merely successively subtracting this constant value of d_x from 100,000.

Just a century later, in 1825, Benjamin Gompertz invented a formula which proceeded on the basis that there was with increasing age "an increasing inability to withstand destruction." This assumed that the force of mortality increased in geometrical progression. That is, in algebraic form,

$$\mu_x = Bc^x$$

in which formula B represents a basic constant number and c the constant number involved in the geometrical increment, which is raised to a power corresponding to the age attained. That is, concretely,

$$\mu_{30} = Bc^{30}$$

Gompertz, in the reasoning which led up to this formula, supplied the basis for a much better formula, though he did not apply it. For he said: "It is possible that death may be the consequence of two generally co-existing causes; the one, chance, without previous disposition to death or deterioration; the other, a deterioration or increased inability to withstand destruction." In his formula he deals with the force of mortality as if the latter of these propositions were alone true; whereas, in fact, they are in all probability equally true.

In 1860 the eminent actuary, Makeham, corrected and improved Gompertz's formula by adding a quantity to represent this constant element in the force of mortality, which made the formula take this form:

$\mu_x = A + Bc^x$, or, for a concrete example,

$\mu_{30} = A + Bc^{30}$

To deal directly with l_x, let $k = l_x$, where x is the lowest age in the table—in other words, let k be the radix of the table.

By De Moivre's hypothesis the deaths at each age would be a constant number. If Δ represent this number, then,

$$l_{x+n} = k - n\Delta = l_{x+n-1} - \Delta.$$

By Gompertz's hypothesis,

$$l_{x+n} = k\, g^{c^{x+n}}$$

To find the rate of increment—it is really a rate of decrement—of this geometric series, recourse must be had to logarithms, thus:

$$l_{x+n} = k\, g^{c^{x+n}}$$
$$l_{x+n-1} = k\, g^{c^{x+n-1}}$$

$$(1.)\quad \text{Log } l_{x+n} = \log k + c^{x+n} \log g$$
$$\text{Log } l_{x+n-1} = \log k + c^{x+n-1} \log g$$

$$(2.)\quad \text{Log } l_{x+n} - \log l_{x+n-1} = (c^{x+n} - c^{x+n-1}) \log g$$
$$= c^{x+n-1}(c - 1) \log g.$$

Also:

$$l_{x+n+1} = k\, g^{c^{x+n+1}}$$
$$l_{x+n} = k\, g^{c^{x+n}}$$
$$\text{Log } l_{x+n+1} = \log k + c^{x+n+1} \log g$$
$$\text{Log } l_{x+n} = \log k + c^{x+n} \log g$$

$$(3.)\quad \text{Log } l_{x+n+1} - \log l_{x+n} = (c^{x+n+1} - c^{x+n}) \log g$$
$$= c^{x+n}(c - 1) \log g.$$

Divide equation (3) by equation (·) and we get:

$$c = \frac{\text{Log } l_{x+n+1} - \log l_{x+n}}{\text{Log } l_{x+n} - \log l_{x+n-1}}$$

And from equation (2) by transposition and division:

$$\text{Log } g = \frac{\text{Log } l_{x+n} - \log l_{x+n-1}}{c^{x+n-1}(c-1)}$$

And from equation (1) by transposition.

$$\text{Log } k = \log l_{x+n} - c^{x+n} \log g.$$

From these values, the value of l_x for any age may be found by the formula:

$$\text{Log } l_x = \log k + c^x \log g.$$

In the foregoing demonstration the constants k, c, and g are derived from three adjacent values of l_x, namely l_{x+n-1}, l_{x+n}, and l_{x+n+1}. If the table of l_x from beginning to end already follow Gompertz's law this method will be entirely satisfactory. Most frequently, however, the formula is applied to rough material, and the use of three adjacent values would give results which would not apply to the rest of the table. To overcome this objection, values of l_x at considerable intervals may be used in the same manner as is outlined above. Thus the values might be taken l_x, l_{x+n} and l_{x+2n}, such as l_{20}, l_{45}, and l_{70} so as to cover a large section of the table; or, if the experience be more complete at the central ages of the table, the values l_{30}, l_{40} and l_{50} might be used. This method of graduation is now seldom employed, as it has been superseded by Makeham's, which is found to adhere more closely to the actual mortality experienced.

Makeham's modification of Gompertz's method gives as the value of l_{x+n}:

$$l_{x+n} = k\, s^{x+n}\, g^{c^{x+n}}$$

Again resorting to logarithms, we have for l_x, l_{x+n}, l_{x+2n}, and l_{x+3n}, respectively:

(4.) $\text{Log } l_x = \log k + x \log s + c^x \log g.$

$\text{Log } l_{x+n} = \log k + (x+n) \log s + c^{x+n} \log g.$

$\text{Log } l_{x+2n} = \log k + (x+2n) \log s + c^{x+2n} \log g.$

$\text{Log } l_{x+3n} = \log k + (x+3n) \log s + c^{x+3n} \log g.$

Subtracting each equation from the one below, we have:

(5.) $\text{Log } l_{x+n} - \log l_x = n \log s + c^x (c^n - 1) \log g$

$\text{Log } l_{x+2n} - \log l_{x+n} = n \log s + c^{x+n} (c^n - 1) \log g.$

$$\text{Log } l_{x+3n} - \log l_{x+2n} = n \log s + c^{x+2n}(c^n - 1) \log g.$$

Again subtracting each equation from the one below and transposing, we have:

(6.) $c^x (c^n - 1)^2 \log g = \text{Log } l_{x+2n} - 2 \log l_{x+n} + \log l_x.$

(7.) $c^{x+n} (c^n - 1)^2 \log g = \text{Log } l_{x+3n} - 2 \log l_{x+2n} + \log l_{x+n}$

Dividing equation (7) by equation (6), we get:

$$c^n = \frac{\text{Log } l_{x+3n} - 2 \log l_{x+2n} + \log l_{x+n}}{\text{Log } l_{x+2n} - 2 \log l_{x+n} + \log l_x}.$$

$$n \log c = \text{Log} \left\{ \frac{\log l_{x+3n} - 2 \log l_{x+2n} + \log l_{x+n}}{\log l_{x+2n} - 2 \log l_{x+n} + \log l_x} \right\}$$

$$\text{Log } c = \frac{1}{n} \text{Log} \left\{ \frac{\log l_{x+3n} - 2 \log l_{x+2n} + \log l_{x+n}}{\log l_{x+2n} - 2 \log l_{x+n} + \log l_x} \right\}.$$

From equation (6) we get:

$$\text{Log } g = \frac{\text{Log } l_{x+2n} - 2 \log l_{x+n} + \log l_x}{c^x (c^n - 1)^2}$$

From equation (5) by transposition and division

$$\text{Log } s = \frac{\text{Log } l_{x+n} - \log l_x - c^x (c^n - 1) \log g}{n}$$

From equation (4) by transposition:

$\text{Log } k = \log l_x - x \log s - c^x \log g.$

The values of k, s, g and c, then, may be determined from four values of l, taken from the unadjusted table at equal intervals—such as l_{20}, l_{40}, l_{60} and l_{80}, in which case the value 20 is given to n in the formulas; or from l_{20}, l_{35}, l_{50} and l_{65}, in which case $n = 15$

The formula for the adjusted value of l_x at all ages is:

$$\text{Log } l_x = \log k + x \log s + c^x \log g\cdot$$

In dealing with "average" lives as in a population experience or an experience involving "ultimate" probabilities, drawn from insured lives—i e., probabilities from lives where the benefit of fresh selection has worn off—Makeham's method applies usually from childhood to the oldest ages. Not so for the younger ages in an "aggregate" experience with insured lives; nor as to lives affected by fresh medical selection in a "select" experience, arranged by years of insurance as well as by ages of life. But the method answers equally well for the older ages in almost any aggregate experience; for there the experience is not affected by fresh selection in any appreciable degree.

There are more suitable methods of applying Makeham's law to the graduation of a mortality table, but they do not come properly within the scope of this book. The most satisfactory method is by employing groups of ages as described in the Institute Text-Book, Part II, the mathematical portion of the work being undertaken either as therein illustrated, or by Dr. Karl Pearson's "method of moments."

THE FORCE OF MORTALITY EMPLOYED IN GRADUATION.

We have already made use of the expression, "force of mortality," μ_x, without attempting to assign it a definite meaning. It is not the same as the "rate of mortality" or probability of dying which we designate by q_x and which is the quotient of the number living at the beginning of the year into the number dying during the year, $\frac{d_x}{l_x}$. The liability to die must be taken as increasing not in broken steps once a year, but constantly and, consequently, the real force of mortality is constantly increasing. The force of mortality is measured by the same denominator or divisor, l_x, as is q_x, but the numerator is not the number who die out of the number beginning the year, but the number who, by the same force of mortality, would have died had the number living been kept full at l_x by adding new lives of the same age and conditions exactly whenever lives failed. This numerator would commonly be somewhat larger than d_x; and at some of the higher ages might be considerably larger. The force of mortality is a mathematical expression which accurately measures the curve in the first illustration of the graphic method in these lessons. We express it by the Greex μ_x, which should not be confused with m_x, which is the central death rate.

It is not possible to get a simple expression of the value of μ_x in terms of l_x, d_x, p_x or q_x without making use of the calculus. An approximate expression, which is said by the Institute of Actuaries' Text-Book to be "for practical purposes usually sufficiently accurate," may be derived by dividing the number dying for the previous and present years by double the number surviving the previous year and setting out upon the present year:

$$\mu_x = \frac{d_{x-1} + d_x}{2\, l_x} = \frac{l_{x-1} - l_{x+1}}{2\, l_x}.$$

In considering probabilities derived directly from population statistics, we have already found something very similar to $_x$ in m_x, the central death-rate. This was long known as the "force of mortality" before Gompertz's and Makeham's labors developed a mathematical law. The name was first mentioned by Thomas R. Edmonds, who also sought for a law, but found nothing suitable, except by arbitrarily altering the constants.

The population given in census statistics at any age is approximately the number surviving at one-half year beyond the birthday; but the deaths given for the same age are, on the other hand, the deaths from birthday to birthday. From which it happens that the quotient obtained by dividing the deaths by the population is not $q_{x+\frac{1}{2}}$ but m_x, thus:

$$m_x = \frac{d_x}{l_{x+\frac{1}{2}}}.$$

If we extend this system so as to cover two years instead of one, we shall produce the approximate formula:

$$\mu_x = \frac{l_{x-1} - l_{x+1}}{2\, l_x}.$$

Two other approximate formulas for μ_x have been developed by central finite differences—generally more nearly accurate than the one already given and equally useful. It will not be here attempted to show the mode of developing them, but they are as follows.

$$\mu_x = \frac{7\,(d_{x-1} + d_x) - (d_{x-2} + d_{x+1})}{12\, l_x},$$

$$\mu_x = \frac{8\,(l_{x-1} - l_{x+1}) - (l_{x-2} - l_{x+2})}{12\,l_x}.$$

When a mortality table does not follow Gompertz's or Makeham's law, nor any other mathematical law, it will be found that there is no expression for the force of mortality that can be relied upon for all ages,

It has been explained that $\mu_x = Bc^x$ by Gompertz's law and $\mu_x = A + Bc^x$ by Makeham's modification of Gompertz's law. The mode of ascertaining the value of A and of B involves the differential calculus and will not be explained here; but the results expressed in constants, the method to obtain which was shown in the last chapter, are as follows:

By Gompertz's law $\mu_x = -$ (log. g log. c) c^x

By Makeham's modification $\mu_x = -\{$ log. s + (log. g log. c) $c^x \}$.

The values of μ_x for any age, either fractional or integral may, then, be accurately ascertained from the values of g and c, when graduating by Gompertz's law and by g, c and s, when graduating by Makeham's process.

The following formula is approximate:

$$m_x = \mu_{x+\frac{1}{2}}.$$

From which the approximate value of m_x:

When by Gompertz's law $m_x = -$ (log. g log. c) $c^{x+\frac{1}{2}}$

When by Makeham's process $m_x = - \{$ log. s + (log. g log. c) $c^{x+\frac{1}{2}} \}$

From a complete series of these values for all ages, a series of values of p_x for all ages could be found by the formula:

$$p_x = \frac{2 - m_x}{2 + m_x}.$$

And a complete mortality table by starting with any radix and using the formula of construction:

$$l_{x+1} = l_x p_x.$$

INTERPOLATION AND GRADUATION BY INTERPOLATION.

In several recent lessons there have been references to completing the outlines of a table, values at certain intervals being determined by one of the systems described, through the interpolation of the intermediate values. In the description of a system of graduating by finite differences, the work was really a continuous interpolation conducted by developing the law governing the series of differences. But this was carefully kept out of the range of the calculus of finite differences and within the comprehension of those who are familiar with ordinary algebra.

A merely rudimentary knowledge of this calculus is, however, so useful in every day calculations by an actuary that we will seek to develop a few simple principles and formulas.

Suppose a series of values of u, such as u_0, u_1, u_2, u_3, and so on. The difference between the first and the second, that is, $u_1 - u_0$, we call Δu_0; the difference between the second and third, or $u_2 - u_1$, is Δu_1, etc. Then for the difference between these differences we have $\Delta^2 u_0$, etc., thus:

Series	Δ	Δ^2	Δ^3	Δ^4	Δ^5, etc.
u_0	Δu_0	$\Delta^2 u_0$	$\Delta^3 u_0$	$\Delta^4 u_0$	$\Delta^5 u_0$
u_1	Δu_1	$\Delta^2 u_1$	$\Delta^3 u_1$	$\Delta^4 u_1$	
u_2	Δu_2	$\Delta^2 u_2$	$\Delta^3 u_2$		
u_3	Δu_3	$\Delta^2 u_3$			
u_4	Δu_4				
u_5					

If we have the value of u_0 and of all the differences bearing the notation u_0, we can reconstruct the entire series for $\Delta^5 u_0 + \Delta^4 u_0 = \Delta^4 u_1$; $\Delta^4 u_0 + \Delta^3 u_0 = \Delta^3 u_1$ and $\Delta^3 u_1 + \Delta^4 u_1 = \Delta^3 u_2$; and so on.

We find the following values of Δ, Δ^2, Δ^3, etc., in terms of u_0, u_1, u_2, etc.:

$\Delta u_0 = u_1 - u_0$ $\quad \Delta^2 u_0 = \Delta u_1 - \Delta u_0 = u_2 - 2u_1 + u_0$
$\Delta u_1 = u_2 - u_1$ $\quad \Delta^2 u_1 = \Delta u_2 - \Delta u_1 = u_3 - 2u_2 + u_1$
$\Delta u_2 = u_3 - u_2$ $\quad \Delta^3 u_0 = \Delta^2 u_1 - \Delta^2 u_0 = u_3 - 3u_2 + 3u_1 - u_0$

It will be observed that this series of values of Δ follow the well-known law of the binomial theorem, which is also readily proven to hold good through all orders of differences. We find another correspondence with the binomial theorem in the following development of the values of u_x in earlier values of u plus differences:

$u_1 = u_0 + \Delta u_0$
$u_2 = u_1 + \Delta u_1 = (u_0 + \Delta u_0) + (\Delta u_0 + \Delta^2 u_0) = u_0 + 2\Delta u_0 + \Delta^2 u_0$
$u_3 = u_2 + \Delta u_2 = (u_0 + 2\Delta u_0 + \Delta^2 u_0) + \Delta(u_0 + 2\Delta u_0 + \Delta^2 u_0)$
$\quad = u_0 + 3\Delta u_0 + 3\Delta^2 u_0 + \Delta^3 u_0$.

This is also readily proven to hold good for all values of u and may be stated generally:

$$u_n = u_0 + n\Delta u_0 + \frac{n(n-1)}{\underline{|2}}\Delta^2 u_0 + \frac{n(n-1)(n-2)}{\underline{|3}}\Delta^3 u_0 + \text{etc.}$$

An adjustment or graduation of a mortality table may be made by taking the values of l at five year intervals such as ten, fifteen, twenty, etc., and interpolating to get the intermediate values. The foregoing formula may be employed to discover these values. Let u_n be the desired value of l, u_1 a value of l (to be known as l_x), at the nearest quinqennial period, which is z years younger or older than the age at which the value is desired. z of course is some number of years less than five, usually not more than two, as values further from l_x than $l_{x\pm 2}$ can best be derived from a higher or lower triad of quinquennial values.

Then we have $u_n = l_x$; $u_1 = l_s$; $u_0 = l_{s-5}$; $u_2 = l_{s+5}$; $\Delta u_0 = l_s - l_{s-5}$; $\Delta u_1 = l_{s+5} - l_s$; and $\Delta^2 u_0 = (l_{s+5} - l_s) - (l_s - l_{s-5}) = l_{s+5} - 2l_s + l_{s-5}$.

Inserting these values in the general formula and stopping at second differences, we have:

$$l_x = l_{s-5} + n\,(l_s - l_{s-5}) + \frac{n(n-1)}{2}(l_{s+5} - 2l_s + l_{s-5})$$

$$= \left[1 - n + \frac{n(n-1)}{2}\right] l_{s-5} + \left[n - n(n-1)\right] l_s + \left[\frac{n(n-1)}{2}\right] l_{s+5}.$$

The interval which in u_0, u_1, u_2, etc., is expressed by unity is here expressed by 5. The interval n is greater or less than 5 by z, that is, $n = 5 \pm z$, accordingly as l_x is below or above l_s. Let us treat z as subtracted from 5 but as being itself a minus or plus quantity according as l_x is below or above l_s. Now when l_x is beyond l_s and the interval between u_0 and u_n is consequently greater than 5, it so happens that z will be a minus quantity. Thus if u_0 be l_{10}, u_1 l_{15} and u_2 l_{20} and u_n be l_{16}, then the interval is 6 and the value of z is -1 so that $16 + z = 16 + (-1) = 15$. To get an expression for this interval between u_0 and u_n, then we must deduct z from 5, which in the foregoing case will give $5 - z = 5 - (-1) = 5 + 1 = 6$. And this will work equally well in cases where l_x is before l_s as, for instance, l_{14}, while l_s is l_{15}. The value of z is then positive, thus $+1$, so that $14 + (+1) = 15$; and the interval is $5 - (+1) = 4$. But the interval in terms of u_0, u_1, u_2, etc., can be but fractional with 5 for a denominator. Thus the formula becomes:

$$n = \frac{5 - z}{5} = 1 - \frac{z}{5}.$$

Inserting this value in our equation, it becomes:

$$l_x = \left[1 - \left(1 - \frac{z}{5}\right) + \frac{\left(1 - \frac{z}{5}\right)\left(1 - \frac{z}{5} - 1\right)}{2}\right] l_{a-5} + \left[1 - \frac{z}{5} - \left(1 - \frac{z}{5}\right)\left(1 - \frac{z}{5} - 1\right)\right] l_a + \left[\frac{\left(1 - \frac{z}{5}\right)\left(1 - \frac{z}{5} - 1\right)}{2}\right] l_{a+5}$$

$$= \left[\frac{z}{5} + \frac{\left(\frac{z}{5}\right)^2 - \frac{z}{5}}{2}\right] l_{a-5} + \left[1 - \left(\frac{z}{5}\right)^2\right] l_a + \left[\frac{\left(\frac{z}{5}\right)^2 - \frac{z}{5}}{2}\right] l_{a+5}$$

$$= \frac{5z + z^2}{50} l_{a-5} + \frac{25 - z^2}{25} l_a + \frac{z^2 - 5z}{50} l_{a+5}$$

$$= \frac{z(5+z)}{50} l_{a-5} + \frac{25 - z^2}{25} l_a - \frac{z(5-z)}{50} l_{a+5}$$

MODIFIED GRADUATION BY INTERPOLATION.
WOOLHOUSE'S FORMULA.

THE last lesson described a system of determining intermediate values of l from quinquennial values taken from an unadjusted table. By this means through continued interpolation a graduated table could be constructed, but it would have many of the faults of the ungraduated. The eminent English actuary, W. S. B. Woolhouse, who was delegated by the Institute of Actuaries and Faculty of Actuaries to adjust the H^m table, invented a system by which these inequalities could be avoided; and the first H^m table was accordingly graduated by his system. This table was afterward largely displaced by a new H^m table graduated according to Gompertz's law and Makeham's formula. The same original adoption of Woolhouse's formula and its displacement by Makeham's may be noted in the case of the French tables recently published.

The basis of Woolhouse's system is finding the arithmetical mean between values of l_x determined by the construction of five different tables by interpolation from five series of quinqennial values of l from the unadjusted table, thus:

$$\begin{array}{llll} l_{10} & l_{15} & l_{20} & l_{25}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ l_{11} & l_{16} & l_{21} & l_{26}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ l_{12} & l_{17} & l_{22} & l_{27}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ l_{13} & l_{18} & l_{23} & l_{28}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ l_{14} & l_{19} & l_{24} & l_{29}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{array}$$

When complete tables have been constructed by interpolation, we will have five different values of l for each age, which we may call l^1, l^2, l^3, l^4 and l^5. Woolhouse's idea is that a very close approximation to the true value may be made by adding these and dividing by 5, thus:

$$l = \frac{l^1 + l^2 + l^3 + l^4 + l^5}{5}$$

This would, as the author of the plan remarks, be a very laborious and circuitous route; and a shorter one was devised, as follows:

Employing the following formulas for interpolation:

$$l_x = \frac{z(5+z)}{50} l_{x-5} + \frac{25 - z^2}{25} l_x - z \frac{(5-z)}{50} l_{x+5}$$

and putting z = —2, —1, 0, + 1 and + 2, successively, we obtain the five different values of l_x thus:

$$\begin{aligned}
l_x &= -.12\, l_{x-7} + .84\, l_{x-2} + .28\, l_{x+3}.\\
l_x &= -.08\, l_{x-6} + .96\, l_{x-1} + .12\, l_{x+4}.\\
l_x &= l_x\\
l_x &= .12\, l_{x-4} + .96\, l_{x+1} - 08\, l_{x+6}\\
l_x &= .28\, l_{x-3} + .84\, l_{x-2} - .12\, l_{x+7}
\end{aligned}$$

$$\begin{aligned}
5\, l_x = &- 12\, l_{x-7} - 08\, l_{x-6} + .12\, l_{x-4} + .28\, l_{x-3}\\
&+ .84\, l_{x-2} + 96\, l_{x-1} + l_x + .96\, l_{x+1}\\
&+ .84\, l_{x+2} + .28\, l_{x+3} + 12\, l_{x+4} - 08\, l_{x+6} - 12\, l_{x+7}.\\
l_x = &-.024\, l_{x-7} - .016\, l_{x-6} + .024\, l_{x-4} + .056\, l_{x-3}\\
&+ .168\, l_{x-2} + .192\, l_{x-1} + .2\, l_x + .192\, l_{x+1}\\
&+ .168\, l_{x+2} + .056\, l_{x+3} + .024\, l_{x+4} - 016\, l_{x+6} - .024\, l_{x+7}.
\end{aligned}$$

Even in this form, the process is cumbrous.

Since a mortality table may have any radix, let us multiply both sides of the equation by 125 and take 125 l_x for a new l_x. Thus we clear of fractions and obtain:

$$\begin{aligned}
l_x = &- 3\, l_{x-7} - 2\, l_{x-6} + 3\, l_{x-4} + 7\, l_{x-3}\\
&+ 21\, l_{x-2} + 24\, l_{x-1} + 25\, l_x + 24\, l_{x+1}\\
&+ 21\, l_{x+2} + 7\, l_{x+3} + 3\, l_{x+4} - 2\, l_{x+6} - 3\, l_{x+7}.
\end{aligned}$$

It should be observed that there are no values of l_{x-5} and l_{x+5}; which signifies that the coefficient of each of these is zero.

Thomas G. Ackland discovered that in this latest form, the value of l_x may be had by the following process:

$$l_x = \Sigma^3_5 + 9\, \Sigma_5 - 3\, \Sigma_{15}$$

By this is meant: Add together the five unadjusted values having l_x for the middle value for every age, x, and set the sums opposite x in each case. Repeat this process twice, making three times in all. To this final sum opposite x, add nine times the first sum in fives and subtract three times the sum of the fifteen unadjusted values of l_x, having l_x as the central value. The result will be the adjusted value of l_x, upon a radix 125 times as large as the radix of the unadjusted table.

The proof of this may by diagram be shown as follows:

	1														First summation in fives.
		1													
			1												
				1											
					1										
	1	1	1	1	1										Second summation in fives.
		1	1	1	1	1									
			1	1	1	1	1								
				1	1	1	1	1							
					1	1	1	1	1						
	1	2	3	4	5	4	3	2	1						Third summation in fives.
		1	2	3	4	5	4	3	2	1					
			1	2	3	4	5	4	3	2	1				
				1	2	3	4	5	4	3	2	1			
					1	2	3	4	5	4	3	2	1		
	1	3	6	10	15	18	19	18	15	10	6	3	1		
					9	9	9	9	9						Add $9\,\Sigma_5$
	1	3	6	10	24	27	28	27	24	10	6	3	1		
3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	Subtract $3\,\Sigma_{15}$
−3	−2	0	3	7	21	24	25	24	21	7	3	0	−2	−3	= Coefficients.

These are the cofficients of the formula.

The following illustrates the work:

AGE.	Unadjusted	Σ^1_5	Σ^2_5	Σ^3_5	Σ^1_{15}
20	10,000				
21	9.947				
22	9,914	49,535			
23	9,859	49,298			
24	9,815	49.056	245.224		
25	9,763	48,793	243,981		
26	9,705	48,542	242,729	1,213,678	
27	9,651	48,292	241,480	1,207,521	144,929
28	9,608	48,046	240,264	1,201,417	144,190
29	9,565	47,807	239,067	1,195,365	
30	9,517	47,577	237,877		
31	9,466	47,345	236,677		
32	9,421	47,102			
33	9,376	46.846			
34	9,322				
35	9,261				

	For age 27.	For age 28.
Σ^3_5	= 1,207,521	= 1,201,417
$9\,\Sigma^1_5$	= 434.628	432.414
	1,642,149	1,633,831
$3\,\Sigma^1_{15}$	= 434,787	432.570
Adjusted	l_{27} = 1,207,362	l_{28} = 1,201,261

To reduce to a radix of 10,000 at age 20, divide by 125; to a radix of 100,000, divide by 12.5.

CONSTRUCTING A SELECT TABLE.

A select table represents the mortality, according both to age and to duration of the insurance. The notation as to such a table is the same as is employed in dealing with aggregate and ultimate tables, excepting that the expression for the age at entry is enclosed in brackets. Thus $l_{[x-4]+4}$ means the number living at age attained x out of $l_{[x-4]}$, the number living at age of entry x—4, four years previous; and $l_{[x]+4}$ means the number living at age attained x + 4 out of $l_{[x]}$ admitted four years before at age x. The functions d, p and q are similarly altered. Thus $d_{[x-2]+2}$ is the number dying at attained age x out of a group, $l_{[x-2]}$, admitted at age x—2.

Thus, in such a table there will be as many values for q_x as there are years of insurance that are deemed to be affected by selection; i. e. $q_{[x]}$ = rate of mortality on new entrants at age x; $q_{[x-1]+1}$ = rate of mortality on lives now aged x, admitted at age x—1; and so on.

In the select tables, constructed by Dr. T. B. Sprague, who introduced such tables in practical form from the experience embraced in the H^m statistics, it was assumed that the benefit to the insurance companies of the selection of applicants wore off by the end of five years; and, consequently, that an ultimate table could be employed for all lives that were insured longer than five years. In the new select tables, drawn from the experience of the British companies from 1863-1893, the select functions have been tabulated for the period of ten years.

The mode of construction usually employed is as follows:

Firstly, construct an "ultimate" table from the experience of the lives which are considered to have passed the period during which the benefit of selection endures, by the usual methods, adopting any convenient radix.

Secondly, draw off and graduate a series of $q_{[x]}$ for all values of x, a series of $q_{[x]+1}$, etc., until all the years affected by selection have been thus taken care of. This graduation may be by an extended form of Makeham's modification of Gompertz' law—this has been tried with some success—by the graphic method, or by finite differences. The graphic method is generally considered most suitable. From these values of $q_{[x]}$, $q_{[x]+1}$, etc., compute corresponding values of $p_{[x]}$, $p_{[x]+1}$, etc.

Thirdly, join the select part of the table on the ultimate table as follows: l_{x+5} represents the number living at age x+5 by the ultimate table; and, for convenience, assume the period affected by selection to be five years. We must consider l_{x+5} then, to be the number yet surviving at age x+5 out of a group of $l_{[x]}$ entrants at age x five years earlier, which group has been diminishing at the successive ratios, $p_{[x]}$, $p_{[x]+1}$, $p_{[x]+2}$, $p_{[x]+3}$ and $p_{[x]+4}$. We know the value of l_{x+5} which has been found by graduating the ultimate table; we wish to find the values of $l_{[x]+4}$, $l_{[x]+3}$, $l_{[x]+2}$, $l_{[x]+1}$ and $l_{[x]}$, which is done as follows:

$$l_{[x]+4} = \frac{l_{x+5}}{p_{[x]+4}}$$

$$l_{[x]+3} = \frac{l_{[x]+4}}{p_{[x]+3}}$$

$$l_{[x]+2} = \frac{l_{[x]+3}}{p_{[x]+2}}$$

$$l_{[x]+1} = \frac{l_{[x]+2}}{p_{[x]+1}}$$

$$l_{[x]} = \frac{l_{[x]+1}}{p_{[x]}}$$

The result is a mortality table, corresponding to the experience under a group of $l_{[x]}$ lives, admitted at age x, differing from the ultimate table from ages x to x+4, but from age x+5 agreeing with it.

The entire select table comprises such a table for each age of entry and all durations and may take the following form:

AGE.	$l_{[x]}$	$l_{[x-1]+1}$	$l_{[x-2]+2}$	$l_{[x-3]+3}$	$l_{[x-4]+4}$	l_x
20..........	956,418					
21..........	945,306	951,636				
22..........	934,548	940,704	943,500			
23..........	924,270	930,110	932,948	933,800		
24..........	914,390	919,975	922,694	923,598	923,800	
25..........	904,880	910,218	912,860	913,698	913,800	913,800
26..........	895,828	900,808	903,360	904,200	904,200	904,200

The meaning is that out of 956,418 entrants at age 20, 951,636 survive one year, 943,500 two years, 933,800 three years, 923,800 four years, 913,800 five years, 904,200 six years, etc.

The same information may be tabulated a little differently in a form recently adopted in the new British Experience Tables, namely:

AGE [x]	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$	$l_{[x]+3}$	$l_{[x]+4}$	l_{x+5}	AGE x+5
20	956,418	951,636	943,500	933,800	923,800	913,800	25
21	945,306	940,704	932,948	923,598	913,800	904,200	26
22	934,548	930,110	922,694	913,698	904,200		27
23	924,270	919,975	912,860	904,200			28
24	914,390	910,218	903,360				29
25	904,880	900,808					30
26	895,828						31

TESTING THE ACCURACY OF GRADUATED TABLES OF MORTALITY.

THE graduation of mortality tables is, strictly speaking, not construction, but modification. The trend of mortality is indicated by the statistical data drawn from actual mortality experience. From this actual mortality which deals with numbers of lives exposed at certain ages and those of them failing and surviving, respectively, a table of values of p_x and q_x may be directly derived. But, when the data are not very numerous, the tables will be found to present many irregularities which may be adjusted. This adjustment is the office of graduation.

But the adjusted or graduated experience must be such as to adequately and fairly account for the deaths and survivals according to the original data, taken as a whole, the idea being merely that surplusage in one portion will fill up discrepancies in another portion. But the graduated tables of values of q_x, for instance, should not, when applied to the numbers of lives exposed at each of several adjoining ages, result in a widening discrepancy. That would indicate that the graduation was something more than a mere adjustment; that it was, in fact, a complete departure from the data from which it was constructed, and proceeded by some other, quite different law. It follows that the graduated ratios should, when applied to original data, not only account approximately for the aggregate deaths, but also that, within a small number of ages, it should also account for the actual deaths with fair accuracy.

For the purpose of testing the success of graduation, therefore, reference must be had back to the original data. A table must be made showing the number of lives exposed to risk and the number failing for each age. The number out of those exposed that would have failed according to the graduated table of values of q_x must next be computed for each age. This is done by multiplying the number exposed at any given age by the value of q_x for that age in the graduated table. The number that should have failed is then set opposite the number which did fail and the error, plus or minus, noted in an adjoining column. The accumulated error is carried forward in another column, so that any wide departure from the original data appears at once and so that also the proper tendency of the graduation to merely smooth out irregularities is manifest if it be successful.

The following table is an example of this system of testing the accuracy of graduation. It has historical interest, also, as being the test of the success of the graduation of the famous H^m tables according to Makeham's formula by comparing it with the original data. It is taken from the Institute of Actuaries' Text-Book, pages 94 and 95:

AGE.	Exposed to Risk.	Actual Deaths.	Expected Deaths.	Error.	Accumulated Error.
10	379	3	2	—1	—1
11	434	0	2	2	1
12	491.5	2	2	0	1
13	578	2	2	0	1
14	731	3	3	0	1
15	908	2	3	1	2
16	1129	0	4	4	6
17	1421	6	6	0	6
18	1810.5	11	9	—2	4
19	2414	17	13	—4	0
20	3293.5	19	19	0	0
21	4578.5	32	28	—4	—4
22	6397	40	41	1	—3

AGE.	Exposed to Risk.	Actual Deaths.	Expected Deaths.	Error.	Accumulated Error.
23.............	8534	66	57	—9	—12
24.............	10936	75	76	1	—11
25.............	13622.5	70	96	26	15
26.............	16339	113	118	5	20
27.............	19170.5	124	140	16	36
28.............	21837	171	163	—8	28
29.............	24588	181	187	6	34
30.............	27112.5	224	209	—15	19
31.............	29213	215	230	15	34
32.............	31232	260	251	—9	25
33.............	32969	274	271	—3	22
34.............	34535.5	300	290	—10	12
35.............	35818.5	295	309	14	26
36.............	36840.5	326	326	0	26
37.............	37360	357	340	—17	9
38.............	37804.5	389	354	—35	—26
39.............	38112.5	405	369	—36	—62
40.............	38195	377	382	5	—57
41.............	37838	396	393	—3	—60
42.............	37258.5	399	403	4	—56
43.......... ...	36534.5	387	410	23	—33
44.............	35693	421	418	—3	—36
45.	34735.5	429	425	—4	—40
46.............	33660.5	421	431	10	—30
47.............	32502	460	437	—23	—53
48.............	31228	440	442	2	—51
49.............	30055.5	459	448	—11	—62
50.............	28855.5	476	454	—22	—84
51.............	27510.5	479	458	—21	—105
52.............	26208.5	446	462	16	—89
53.............	24785	426	464	38	—51
54.............	23426	444	467	23	—28
55.............	22170.5	509	471	—38	—66
56.............	20746	479	470	—9	—75
57.........	19377.5	463	469	6	—69
58.............	18116.5	455	470	15	—54
59.............	16890.5	428	469	41	—13
60.............	15672.5	488	468	—20	—33
61.............	14392.5	468	461	—7	—40

AGE.	Exposed to Risk.	Actual Deaths.	Expected Deaths.	Error.	Accumulated Error.
62.............	13261	459	458	—1	—41
63.............	12147.5	454	452	—2	—43
64.......... ...	11021.5	443	442	—1	—44
65.............	9984.5	435	432	—3	—47
66......	9009.5	421	421	0	—47
67.............	8081	396	408	12	—35
68.............	7214	399	394	—5	—40
69.............	6375.5	389	377	—12	—52
70.............	5622	315	360	45	—7
71.............	4953	308	344	36	29
72.............	4378	349	330	—19	10
73.............	3771.5	297	308	11	21
74.............	3228	340	286	—54	—33
75.............	2693	254	259	5	—28
76.............	2253	240	235	—5	—33
77.............	1848.5	201	209	8	—25
78.............	1531	188	188	0	—25
79.............	1257	171	167	—4	—29
80.............	995	140	144	4	—25
81.............	782	125	122	—3	—28
82.............	609.5	105	103	—2	—30
83.............	464	96	85	—11	—41
84.............	339	61	67	6	—35
85.............	254.5	55	55	0	—35
86.............	184	40	43	3	—32
87.............	128.5	28	32	4	—28
88.............	91.5	26	25	—1	—29
89.............	57.5	11	17	6	—23
90.............	43.5	10	14	4	—19
91.............	32	10	11	1	—18
92.............	20	9	7	—2	—20
93.............	10.5	7	4	—3	—23
94.............	4	0	2	2	—21
95.............	3.5	1	2	1	—20
96.............	2	2	1	—1	—21
97.............	0	0	0	0	—21
Totals......	1199092.5	20517	20496	422 —443	—21

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LIFE PROBABILITIES.

We have now seen how the probabilities of living ca range themselves into an accurate and regular record (the durations of the lives of a number of persons, in th form of a mortality table. In our consideration of thes probabilities up to this time we have dealt only with deat or survival within one year of age. The probability (surviving one year from the age x we have designate by the expression p_x and the probability of dying by q_x In these expressions x has reference only to the age, an there is no character to designate the time covered, on year being understood wherever no time is designatec Whenever any other period of time is to be designated a n years, it appears as a prefix, thus: ${}_np_x$.

We have found that

$p_x = \frac{l_{x+1}}{l_x}$, and we may now give this the ger eral form:

$${}_np_x = \frac{l_{x+n}}{l_x}.$$

We also found in our study of probabilities that th probability of two or more events happening is measure by the products of their separate probabilities, thus:

$$p^{1,2,3,} \cdot \cdot \cdot \cdot = p^1 \times p^2 \times p^3 \text{, etc.}$$

It follows that the probability that two lives, x and y will survive the term n is expressed by

${}_np_x \times {}_np_y = {}_np_{xy}$, which latter expression wi be hereafter employed.

The expression ${}_np_{xy}$ may be extended in terms of l a follows:

$${}_np_{xy} = {}_np_x \times {}_np_y = \frac{l_{x+n}}{l_x} \times \frac{l_{y+n}}{l_y} = \frac{l_{x+n} \times l_{y+n}}{l_x \times l_y},$$

which latter may be condensed into the expression $\frac{l_{x+n:y+n}}{l_{xy}}$ and still further by using the expression $\frac{{}^{n}l_{xy}}{l_{xy}}$. In this and in all future expressions a prefix in the upper position will indicate that the lives are to be increased in age by that quantity. Thus ${}_{n}^{t}p_{xyz}$ means the probability that three lives aged x + t, y + t and z + t years, respectively, will survive n years.

Likewise we have found that

$$q_x = \frac{d_x}{l_x} = \frac{l_x - l_{x+1}}{l_x} = 1 - p_x .$$

From this we may derive the value of q_x, covering a longer term than one year. This value we will express by $|_n q_x$, a want of uniformity with the expression ${}_np_x$, which appears in the English usage, and has been adopted into general use by the formal acceptance of the English symbols by the last congress of actuaries. In consequence of this the English symbols will be employed throughout this work. Making our formula general, we have

$$|_n q_x = \frac{l_x - l_{x+n}}{l_x} = 1 - {}_np_x$$

The probability that two lives, x and y, will fail, will be $|_n q_x \times |_n q_y$, which we may express by $|_n q_{\overline{xy}}$, the bar indicating we are dealing with the survivor of the two; i. e.—both.

The probability that at least one life will survive is the reverse or complement of the above, namely:

$$\begin{aligned} {}_np_{\overline{xy}} &= 1 - |_n q_{\overline{xy}} = 1 - (1 - {}_np_x)\ (1 - {}_np_y) \\ &= {}_np_x + {}_np_y - {}_np_{xy}. \end{aligned}$$

The probability that x will survive and y die is the multiple of the respective probabilities, viz.:

$${}_np_x\,(1 - {}_np_y) = {}_np_x - {}_np_{xy}.$$

The probability that one will survive and the other die is the above probability, increased by the probability that y will survive and x die, thus:

$$({}_np_x - {}_np_{xy}) + ({}_np_y - {}_np_{xy}) = {}_np_x + {}_np_y - 2\,{}_np_{xy}.$$

The chance that at least one life will fail is the reverse or complement of ${}_np_{xy}$, the probability that both will survive, thus: $|_nq_{xy} = 1 - {}_np_{xy}$.

The probability that x will die in the nth year is the product of the probability that he will survive n — 1 years and the probability that, when he has so survived, he will die in the nth year, thus: ${}_{n-1}p_x \times q_{x+n-1}$; or the same probability may be expressed in terms of p alone. For the probability of death the nth year is clearly the probability of surviving n — 1 years less the probability of surviving one year longer or n years, thus:

$${}_{n-1|}q_x = {}_{n-1}p_x - {}_np_x .$$

Or this probability may be directly derived from the mortality table, by dividing the deaths at age $x + n - 1$ by l_x, thus:

$${}_{n-1|}q_x = \frac{d_{x+n-1}}{l_x}.$$

The probability that both x and y will die in the nth year is the product of the probabilities that each will die, thus:

$${}_{n-1|}q_x \times {}_{n-1|}q_y = ({}_{n-1}p_x - {}_np_x)({}_{n-1}p_y - {}_np_y) = \frac{d_{x+n-1}}{l_x} \times \frac{d_{y+n-1}}{l_y}$$

The probability that the first life of x and y will fail in the nth year is the probability that both will survive n — 1 years less the probability that both will survive n years, thus: ${}_{n-1|}q_{xy} = {}_{n-1}p_{xy} - {}_np_{xy}$.

The probability that the second life of x and y will fail in the nth year is the probability that at least one will survive n — 1 years less the probability that at least one will survive n years, thus:

$$
{}_{n-1}|q_{\overline{xy}} = {}_{n-1}p_{\overline{xy}} - {}_np_{\overline{xy}} = {}_{n-1}p_x + {}_{n-1}p_y - {}_{n-1}p_{xy} - ({}_np_x + {}_np_y - {}_np_{xy}) = {}_{n-1}p_x - {}_np_x + {}_{n-1}p_y - {}_np_y - ({}_{n-1}p_{xy} - {}_np_{xy}) = {}_{n-1}|q_x + {}_{n-1}|q_y - {}_{n-1}|q_{xy}.
$$

The probability that one only of the lives, x and y, will fail in the nth year is the sum of the compound probabilities that x will die in the nth year, and y not die that year, ${}_{n-1}|q_x(1 - {}_{n-1}|q_y)$, and that y will die in the nth year and x not die that year, ${}_{n-1}|q_y(1 - {}_{n-1}|q_x)$, thus:

$$
{}_{n-1}|q_x(1 - {}_{n-1}|q_y) + {}_{n-1}|q_y(1 - {}_{n-1}|q_x) = {}_{n-1}|q_x - {}_{n-1}|q_x({}_{n-1}|q_y) + {}_{n-1}|q_y - {}_{n-1}|q_y({}_{n-1}|q_x) = {}_{n-1}|q_x + {}_{n-1}|q_y - 2({}_{n-1}|q_x)({}_{n-1}|q_y).
$$

The probability that neither of the lives will fail in the nth year is the product of the probabilities that each will not fail, thus:

$$
(1 - {}_{n-1}|q_x)(1 - {}_{n-1}|q_y) = 1 - {}_{n-1}|q_x - {}_{n-1}|q_y + {}_{n-1}|q_x \times {}_{n-1}|q_y.
$$

The probability that one life at least will fail is the reverse or complement of the last foregoing, thus:

$$
1 - (1 - {}_{n-1}|q_x - {}_{n-1}|q_y + {}_{n-1}|q_x \times {}_{n-1}|q_y) = {}_{n-1}|q_x + {}_{n-1}|q_y - {}_{n-1}|q_x \times {}_{n-1}|q_y.
$$

The probability that x will survive n years and y, n — 1 years ~~would~~ be ${}_np_x \times {}_{n-1}p_y$. We may make this analogous to joint life formulas in the following manner: The probability of y surviving n — 1 years is a component part of the probability of y — 1 surviving n years, which latter probability may be stated as first the probability of surviving one year and then of surviving n — 1 years. The probability of y — 1 surviving n years may be thus expressed:

$${}_np_{y-1} = p_{y-1} \times {}_{n-1}p_y,$$ from which

$${}_{n-1}p_y = \frac{{}_np_{y-1}}{p_{y-1}}.$$ Substituting this value in ${}_np_x \times {}_{n-1}p_y$, we have: $$\frac{{}_np_x \times {}_np_{y-1}}{p_{y-1}} = \frac{{}_np_{x:y-1}}{p_{y-1}}$$

EXPECTATION OF LIFE.

THE probable life of an individual in a group such as in group l_x, is until just one-half the group will have died, his chances being even to be among those who have died or those who survive. Probably this should have been called "expectation," but that name was given to something slightly different. The name "expectation" or "expectancy" is given to what might perhaps more accurately be called the "equation of life," i. e., the average number of years men of a given age will survive.

Suppose, then, a group of l_x persons. Of these d_x will die during the year, and, on the hypothesis that deaths are uniform throughout the year, these will have lived in all one-half d_x years. The next year d_{x+1} will die, having survived the first year; these will have lived in all one and one-half d_{x+1} years; and so on. The whole group, then, taken together, will live a series of years, computed as follows: $\frac{1}{2} d_x + \frac{3}{2} d_{x+1} + \frac{5}{2} d_{x+2} + \dots \dots \dots$ or
$\frac{1}{2}(d_x + d_{x+1} + d_{x+2}) \dots + d_{x+1} + 2\, d_{x+2} + 3\, d_{x+3} + \dots$
But the sum of the series $d_x = l_x$; that is, in the end all the group and no more die; and $d_{x+1} + 2\, d_{x+2} + 3\, d_{x+3} + \dots$
breaks up into series, $d_{x+1} + d_{x+2} + d_{x+3} + \dots\dots$
$+ d_{x+2} + d_{x+3} + \dots\dots$
$+ d_{x+3} + \dots\dots$
etc.

which are respectively equal to $l_{x+1} + l_{x+2} + l_{x+3} + \dots$

Therefore the whole series of years takes the form $\frac{1}{2}\, l_x + l_{x+1} + l_{x+2} + l_{x+3} + \dots\dots\dots$

Since this is the whole number of years that all the lives of the group will live, it follows that the average for each will be found by dividing by l_x. This average is called the complete expectation of life and the formula becomes:

$$\mathring{e}_x = \frac{1}{2} + \frac{l_{x+1} + l_{x+2} + l_{x+3} + \dots\dots\dots}{l_x}$$

The curtate expectation of life is the same, less the half-year which on the average the members of the group will live during the year in which death occurs, thus:

$$e_x = \frac{l_{x+1} + l_{x+2} + l_{x+3} + \dots\dots\dots}{l_x}$$

The average for each being $\mathring{e}_x$, it follows that the average age of the individuals of the group at death will be $x + \mathring{e}_x$.

There may be computed also an expectation during the term of n years or after the term of n years, $|_n e_x$ and $_n|e_x$ respectively, whose values are seen to be, using curtate expectations only:

$$|_n e_x = \frac{l_{x+1} + l_{x+2} + \dots\dots\dots + l_{x+n}}{l_x}$$

$$_n|e_x = \frac{l_{x+n+1} + l_{x+n+2} + \dots\dots\dots}{l_x}$$

These two, taken together, are equal to e_x.

To simplify, we may write these formulas as follows:

$$e_x = \frac{\Sigma\, l_{x+1}}{l_x}$$

$$_n|e_x = \frac{\Sigma\, l_{x+n+1}}{l_x}$$

$$|_n e_x = \frac{\Sigma\, l_{x+1} - \Sigma\, l_{x+n+1}}{l_x}$$

We have seen that $\frac{l_{x+1}}{l_x} = p_x$, the probability of surviving one year and generally $\frac{l_{x+n}}{l_x} = {}_n p_x$, the probability of surviving the nth year. Consequently our formula takes the form:

$$e_x = \Sigma \ {}_n p_x$$

The curtate expectation, then, is the sum of the probabilities of surviving each year of life.

From this fact we may proceed to discover the value of the joint expectation of x and y, the average number of years that will pass while both are living. Out of $l_x + l_y$ persons, the total number of pairs of persons, one being aged x and one y, will be $l_x \times l_y$, which we may designate as l_{xy}. In one year the total number will be $l_{x+1} + l_{y+1}$ and the number of pairs $l_{x+1} : {}_{y+1}$, which we may designate as ${}^1l_{xy}$. The total years of joint life, then, will be the sum:

$${}^1l_{xy} + {}^2l_{xy} + {}^3l_{xy} + \ldots\ldots\ldots\ldots$$

The average number of years of joint life or the joint life curtate expectation may be found by dividing by l_{xy}, the original number of pairs, thus:

$$e_{xy} = \frac{{}^1l_{xy} + {}^2l_{xy} + {}^3l_{xy} + \ldots\ldots\ldots\ldots}{l_{xy}}$$

The probability of joint life for one year is clearly $\frac{{}^1l_{xy}}{l_{xy}}$ and generally ${}_n p_{xy} = \frac{{}^n l_{xy}}{l_{xy}}$; so that our formula may be written:

$$e_{xy} = \Sigma \ {}_n p_{xy}$$

The expectation of x after the death of y is the sum of the compound probabilities that x will survive each year and y will not survive each year, thus:

$$e_{y|x} = \Sigma \ {}_n p_x (1 - {}_n p_y) = \Sigma ({}_n p_x - {}_n p_{xy})$$
$$= e_x - e_{xy}$$

PURE ENDOWMENTS AND ANNUITIES.

Suppose persons at age x, equal in number to l_x, according to the mortality tables, and that each wishes to secure \$1 to himself if he survive one year; what equal sum must each pay in advance?

At the end of the year l_{x+1} persons will survive, requiring l_{x+1} dollars. Let the present value of \$1 due in one year be v; then the value of l_{x+1} dollars due in one year is vl_{x+1}. That sum in hand, improved at interest, will amount to l_{x+1} dollars at the end of the year, and is, therefore, the aggregate sum which the l_x persons must pay in advance. As each is to pay the same amount, the quota for each will be $\frac{vl_{x+1}}{l_x}$ dollars.

A promise to pay a sum of money upon the contingency of survival only, is called a pure endowment. The single premium, paid in advance, is designated by the expression ${}_nE_x$, x standing for the age and n for the term; when the term is one year the prefix is omitted. We may thus write:

$$E_x = \frac{vl_{x+1}}{l_x}$$

If ${}_2E_x$ were desired, we should find it by dividing the present value of l_{x+2} dollars by the number of contributors, l_x. But the present value of \$1 due in two years is v^2. So we have

$${}_2E_x = \frac{v^2\, l_{x+2}}{l_x} \text{ and, generally,}$$

$${}_nE_x = \frac{v^n\, l_{x+n}}{l_x}$$

In insurance we shall often use the term "present value," not merely in its ordinary arithmetical sense as above, but also to express the idea of the present value of a sum certain, modified by the probability of securing it. Thus ${}_nE_x$ may be called the present value of a pure endowment for n years from age x; it is the present value of one dollar due in n years, modified by ${}_np_x$ or the probability of surviving n years and may be expressed thus:

$${}_nE_x = v^n {}_np_x .$$

We shall find this to be equivalent to the former expression; for $\frac{l_{x+1}}{l_x} = p_x$, $\frac{l_{x+2}}{l_x} = {}_2p_x$ and, generally, $\frac{l_{x+n}}{l_x} = {}_np_x$. Writing then our former formula and substituting these values, we have:

$${}_nE_x = \frac{v^n l_{x+n}}{l_x} = v^n \frac{l_{x+n}}{l_x} = v^n {}_n p_x .$$

An annuity is a promise to pay a like sum at the end of 1, 2, 3, 4, etc., years; its present value is the present value of the sum due in one year, plus the present value of the sum due in two years, plus etc. The present value of an annuity certain is $v + v^2 + v^3 + v^4 +$ etc.

A life annuity is one in which each payment is contingent on surviving the year; its present value is, therefore, the present value of a pure endowment in one year plus the present value of a pure endowment in two years, plus etc. It may be expressed thus:

$$\begin{aligned} |_n a_x &= E_x + {}_2E_x + {}_3E_x + \dots\dots\dots {}_nE_x \\ &= v p_x + v^2 {}_2p_x + v^3 {}_3p_x + \dots\dots v^n {}_np_x \\ &= \frac{v l_{x+1} + v^2 l_{x+2} + v^3 l_{x+3} + \dots\dots v^n l_{x+n}}{l_x} \end{aligned}$$

Where the term of the annuity is the life of the annuitant, it is called a life annuity; where it is a term of years, conditioned upon survival, it is called a temporary life annuity. The algebraic expression for the present value of a temporary annuity is $|_n a_x$, the prefix indicating the limited term and the suffix the age. The expression for the present value of a life annuity is a_x, the absence of the prefix meaning, not that the term is one year, but that it is for life. To arrive at the present value of any annuity, we must first compute the present value of the simple endowments composing it and must then sum these values.

This summation may be somewhat simplified, however, as follows: When one has survived one year, and the first dollar is due, the value of his annuity is one dollar plus the present value of an annuity at an age one year older than his age at the outset, thus: $1 + a_{x+1}$, in the case of a life annuity. It follows that the value to him at the outset, one year earlier, was the present value of a pure endowment of $1 + a_{x+1}$ dollars, due in one year, or

$$a_x = E_x (1 + a_{x+1}) = v\, p_x (1 + a_{x+1})$$

Starting, therefore, at the extreme age of the mortality table (say 100), we may make a complete table of life annuities, thus, for instance:

$$a_{99} = v\, p_{99}$$
$$a_{98} = v\, p_{98} (1 + a_{99})$$
$$a_{97} = v\, p_{97} (1 + a_{98}), \text{ etc.}$$

NOTE.—This is the English notation throughout. In American notation the symbol for the single premium for an annuity is A_x, which also means an immediate annuity or with the first payment due now and not one year from now. It is equal to $1 + a_x$, English notation.

COMMUTATION COLUMNS N AND D.

In the last paper we derived the following formula for the present value of a life annuity:

$$a_x = \frac{v\,l_{x+1} + v^2\,l_{x+2} + v^3\,l_{x+3},\ \text{etc.}}{l_x}$$

This formula is general in its application, but only as a formula. We have seen that values of *a* have a certain relation so that, one value being given, we may derive the next lower. We also know that the numerator in any value of *a* will consist of values of l for ages higher than the present age of the annuitant, but these values of l will not be modified with v raised to the same power. Thus

$$a_{x+1} = \frac{v\,l_{x+2} + v^2\,l_{x+3} + v^3\,l_{x+4},\ \text{etc.}}{l_{x+1}}$$

$$\text{and not } \frac{v^2\,l_{x+2} + v^3\,l_{x+3} + v^4\,l_{x+4},\ \text{etc.}}{l_{x+1}}$$

In other words the index of v and the suffix of l do not agree at all; we are constantly dealing with the same values of l, but modified by changing powers of v.

But let us multiply the numerator and denominator of the second term of the equation:

$$a_x = \frac{v\,l_{x+1} + v^2\,l_{x+2} + v^3\,l_{x+3},\ \text{etc.}}{l_x}$$

by v^x which will not alter the value; this gives us

$$a_x = \frac{v^{x+1}l_{x+1} + v^{x+2}l_{x+2} + v^{x+3}l_{x+3} + \text{etc.}}{v^x\,l_x}$$

Now let us multiply the numerator and denominator of the second term of the equation:

$$a_{x+1} = \frac{v\,l_{x+2} + v^2\,l_{x+3} + v^3\,l_{x+4} + \text{etc.}}{l_{x+1}}$$

by v^{x+1}, which also will not alter the value. We have

$$a_{x+1} = \frac{v^{x+2}l_{x+2} + v^{x+3}l_{x+3} + v^{x+4}l_{x+4} + \text{etc.}}{v^{x+1}l_{x+1}}$$

The numerator in this formula for the value of a_{x+1} is the same as the numerator in the formula for the value of a_x, except that the first term, $v^{x+1}l_{x+1}$, is omitted; the denominator is the value of l corresponding to the age of the annuitant (x+1), multiplied by v raised to the same power (x+1). And this will be found to be general, thus:

$$a_{x+n} = \frac{v^{x+n+1}\, l_{x+n+1} + v_{x+n+2}\, l_{x+n+2} + \text{etc.}}{v^{x+n}l_{x+n}}$$

It follows that when we have once derived the values $v^x l_x$ for each age, we may employ them to compute annuity values at any age. Let us designate these values as D_x, D_{x+1}, D_{x+2}, etc. We then have

$$a_x = \frac{v^{x+1}l_{x+1} + v^{x+2}l_{x+2} + v^{x+3}l_{x+3} + \text{etc.}}{v^x\, l_x}$$

$$= \frac{D_{x+1} + D_{x+2} + D_{x+3} + \text{etc.}}{D_x}$$

And, generally,

$$a_{x+n} = \frac{D_{x+n+1} + D_{x+n+2} + D_{x+n+3} + \text{etc.}}{D_{x+n}}$$

These are the values in the D column of the commutation tables.

A still greater saving in time and labor is effected by summing, once for all, the values of D above each age, thus furnishing at sight the numerators for the formula as well as the denominators. The sum of the values above age x is called N_x and the formula takes this form

$$a_x = \frac{D_{x+1} + D_{x+2} + D_{x+3} + \text{etc.}}{D_x} = \frac{N_x}{D_x}$$

These sums are the values in the N columns of the commutation tables.

The annuity values with which we have so far dealt, are values of annuities with first payment due in one year; an annuity of one dollar with the first payment due now is worth just one dollar more than this value, thus: $1+a_x$. We may derive its value directly from the commutation tables, thus:

$$1+a_x = 1 + \frac{N_x}{D_x} = 1 + \frac{D_{x+1} + D_{x+2} + D_{x+3} + \text{etc.}}{D_x}$$

$$= \frac{D_x}{D_x} + \frac{D_{x+1} + D_{x+2} + D_{x+3} + \text{etc.}}{D_x}$$

$$= \frac{D_x + D_{x+1} + D_{x+2} + D_{x+3} + \text{etc.}}{D_x}$$

$$= \frac{N_{x-1}}{D_x}$$

Note.—The English notation as agreed upon in the last Congress of Actuaries is used in the foregoing throughout. In American notation, as used in old American publications hitherto, A_x instead of a_x is the symbol for an annuity and, besides, signifies the value of an immediate annuity. It is therefore equivalent to $1+a_x$ in the English notation.

N_x, in the English notation as here adopted, includes only the values of D at ages above x; in the American notation it also includes D_x.

The reason for these differences is that English actuaries first had to do with annuities, first payment due in one year, while American actuaries first had to do with annuity values in connection with insurance premiums in which annuities due are employed.

British actuaries distinguish the initial N by writing it $\mathfrak{N}$ and American actuaries nowadays by using a heavy block-letter $\mathbf{N}$. The initial N is always used in select tables in which the use of $N_{[x]-1}$ for $D_{[x]} + D_{[x]+1+x}$ would be contrary to the principles on which the notation is based.

DEFERRED ANNUITIES.

We have so far dealt with annuities, first payment due in one year with a formula as follows:

$$\begin{aligned} a_x &= {}_1E_x + {}_2E_x + {}_3E_x + \;.\;.\;.\;.\;.\;.\;.\;.\;.\;. \\ &= \frac{v\,l_{x+1} + v^2\,l_{x+2} + v^3\,l_{x+3} + \;.\;.\;.\;.\;.\;.\;.\;.}{l_x} \\ &= \frac{v^{x+1}l_{x+1} + v^{x+2}l_{x+2}\;\; v^{x+3}l_{x+3} + \;.\;.\;.\;.\;.\;.\;.}{v^x\,l_x} \\ &= \frac{D_{x+1} + D_{x+2} + D_{x+3} + \;.\;.\;.\;.\;.\;.\;.\;.\;.}{D_x} \\ &= \frac{N_x}{D_x} \end{aligned}$$

But let it be required to find the value of this same annuity a year earlier or at age x — 1, payments not to begin until age x + 1 as in this annuity. It is evident that the value of such an annuity would be the present value of a pure endowment equal to a_x, thus: ${}_1E_{x-1}a_x$. The value of ${}_1E_{x-1}$ is, by formula, $\frac{v\,l_x}{l_{x-1}}$. Hence

$$\begin{aligned} {}_1E_{x-1}a_x &= \frac{v\,l_x}{l_{x-1}}\;a_x \\ &= \frac{v\,l_x}{l_{x-1}} \times \frac{v\,l_{x+1} + v^2\,l_{x+2} + v^3\,l_{x+3} + \;.\;.\;.\;.\;.}{l_x} \\ &= \frac{v\,(v\,l_{x+1} + v^2\,l_{x+2} + v^3\,l_{x+3} + \;.\;.\;.\;.\;.\;.)}{l_{x-1}} \\ &= \frac{v^2\,l_{x+1} + v^3\,l_{x+2} + v^4\,l_{x+3} + \;.\;.\;.\;.\;.\;.\;.}{l_{x-1}} \end{aligned}$$

In other words the fraction, expressing a_x, is modified by raising the power of v and substituting l_{x-1} for l_x for denominator. The algebraic symbol for a deferred annuity is $_{n|}a_x$; and we may give the formula a general expression:

$$_{n|}a_x = {}_nE_x\, a_{x+n} = \frac{v^{n+1}\, l_{x+n+1} + v^{n+2}\, l_{x+n+2} + \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot}{l_x}$$

It will be observed that a_{x+n} does not signify an annuity with the first payment in n years, but a life annuity deferred n years, the first payment being due one year later or in n + 1 years. Thus the first payment on an annuity deferred twenty years is due not in twenty years but in twenty-one years.

In the foregoing formula we have $_{n|}a_x = {}_nE_x\, a_{x+n}$. We have found the formula: $_nE_x = v^n\, {}_np_x$. Substituting, this gives

$$_{n|}a_x = {}_nE_x\, a_{x+n} = v^n\, {}_np_x\, (a_{x+n}).$$

But we have also seen that $v^n\, {}_np_x = \frac{D_{x+n}}{D_x}$ and that $a_{x+n} = \frac{N_{x+n}}{D_{x+n}}$. Substituting again, we get the general formula:

$$_{n|}a_x = {}_nE_x\, a_{x+n} = v^n\, {}_np_x\, (a_{x+n})$$
$$= \frac{D_{x+n}}{D_x} \times \frac{N_{x+n}}{D_{x+n}} = \frac{N_{x+n}}{D_x}$$

TEMPORARY ANNUITIES.

In our original consideration of annuities, we have a foundation for formulæ for a temporary annuity which may be defined as an annuity limited to a term of years, first payment due in one year. A pure endowment due in one year may be considered a temporary annuity limited to one year. A temporary annuity limited to two years will have a value greater than this by just the value of the additional payment or the pure endowment due in two years. And, generally, adopting the algebraic symbol $|_n a_x$, to express a temporary annuity limited to n years, we have:

$$|_n a_x = {}_1E_x + {}_2E_x + {}_3E_x + \dots\dots\dots \quad {}_nE_x$$

$$= \frac{v\, l_{x+1} + v^2\, l_{x+2} + v^3\, l_{x+3} + \quad \dots \quad v^n\, l_{x+n}}{l_x}$$

It will be observed that this differs from the first two equations of general formula for a life annuity:

$$a_x = {}_1E_x + {}_2E_x + {}_3E_x + \dots\dots\dots\dots$$

$$= \frac{v\, l_{x+1} + v^2\, l_{x+1} + v^3\, l_{x+1} + \dots\dots\dots}{l_x}$$

$$= \frac{v^{x+1} l_{x+1} + v^{x+2} l_{x+2} + v^{x+3} l_{x+3} + \dots\dots}{v^x\, l_x}$$

$$= \frac{D_{x+1} + D_{x+2} + D_{x+3} + \dots\dots\dots}{D_x}$$

$$= \frac{N_x}{D_x}$$

only in the particular that the factors of the numerator stop at ${}_nE_x$ and $v^n\, l_{x+n}$ instead of running to the end of the series. As the second two equations in the formula are equivalent in every detail to the first two, it follows that

$$|_n a_x = \frac{D_{x+1} + D_{x+2} + D_{x+3} + \;.\;.\;.\;.\;.\;.\; D_{x+n}}{D_x}$$

This may be verified by multiplying by v^x the numerator and denominator in the formula:

$$|_n a_x = \frac{v\,l_{x+1} + v^2\,l_{x+2} + v^3\,l_{x+3} \;.\;.\;.\;.\;.\; v^n\,l_{x+n}}{l_x}$$

which does not alter the value and gives

$$|_n a_x = \frac{v^{x+1} l_{x+1} + v^{x+2} l_{x+2} + v^{x+3} l_{x+3} + \;.\; v^{x+n} l_{x+n}}{v^x\,l_x}$$

the identity of which with

$$|_n a_x = \frac{D_{x+1} + D_{x+2} + D_{x+3} + \;.\;.\;.\;.\;.\;.\; D_{x+n}}{D_x}$$

is evident by inspection.

The numerator of this latter fraction is, then, less than N_x by all the values in the series beyond the age $x + n$, that is: by $D_{x+n+1} + D_{x+n+2} + D_{x+n+3} + \;.\;.\;.\;.\;.\;.\;.$ But the value of these is summed in N_{x+n}, whence we have

$$|_n a_x = \frac{D_{x+1} + D_{x+2} + D_{x+3} + \;.\;.\; D_{x+n}}{D_x} = \frac{N_x - N_{x+n}}{D_x}$$

CONVERSION FORMULAS.—LIFE, TEMPORARY AND DEFERRED ANNUITIES.

We have so far considered deferred and temporary annuities in comparison with life annuities only; it remains to compare them with each other. Suppose two annuities, one limited to n years from age x and one deferred n years from age x. We find their values to be as follows:

$$
\begin{aligned}
|_n a_x &= {}_1E_x + {}_2E_x + {}_3E_x + \quad . \; . \; . \; . \; . \; . \; . \; {}_nE_x \\
&= \frac{v\, l_{x+1} + v^2\, l_{x+2} + v^3\, l_{x+3} + \quad . \; . \; . \; v^n\, l_{x+n}}{l_x} \\
&= \frac{D_{x+1} + D_{x+2} + D_{x+3} + \quad . \; . \; . \; . \; D_{x+n}}{D_x} \\
&= \frac{N_x - N_{x+n}}{D_x}
\end{aligned}
$$

$$
\begin{aligned}
\text{and } {}_n|a_x &= {}_nE_x\, a_{x+n} = {}_{n+1}E_x + {}_{n+2}E_x + \; . \; . \; . \; . \; . \\
&= \frac{v^{n+1}\, l_{x+n+1} + v^{n+2}\, l_{x+n+2} + \; . \; . \; . \; . \; .}{l_x} \\
&= \frac{D_{x+n+1} + D_{x+n+2} + \; . \; . \; . \; . \; . \; . \; . \; . \; .}{D_x} \\
&= \frac{N_{x+n}}{D_x}
\end{aligned}
$$

In other words, the value of the deferred annuity is expressed by those factors in the series $v\, l_x$ or the series D_x beyond the factors expressing the value of the temporary annuity. They are thus found to be complementary to each other, their sum being equal to a_x, as we may prove by adding them, thus:

$$
\begin{aligned}
&({}_1E_x + {}_2E_x + {}_3E_x + \; . \; . \; {}_nE_x) + ({}_{n+1}E_x + {}_{n+2}E_x \; . \; .) \\
&= {}_1E_x + {}_2E_x + {}_3E_x \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; .
\end{aligned}
$$

The same result may be expressed thus:

$$\frac{(v\,l_{x+1} + v^2 l_{x+2} + \;.\;.\; v^n\,l_{x+n}) + (v^{n+1} l_{x+n+1} + v^{n+2} l_{x+n+2} + \;.\;.)}{l_x}$$

$$= \frac{v\,l_{x+1} + v^2 l_{x+2} + v^3 l_{x+3} + \;.\;.\;.\;.\;.\;.\;.\;.\;.}{l_x} = a_x\;;$$

$$\frac{(D_{x+1} + D_{x+2} + \;.\;.\; D_{x+n}) + (D_{x+n+1} + D_{x+n+2} + \;.\;.)}{D_x}$$

$$= \frac{D_{x+1} + D_{x+2} + D_{x+3} + \;.\;.\;.\;.\;.\;.\;.\;.\;.}{D_x} = a_x\;;$$

and, plainly, $\frac{(N_x - N_{x+n}) + N_{x+n}}{D_x} = \frac{N_x}{D_x} = a_x$

The values of temporary and deferred annuities are thus seen to be parts of a life annuity, which furnishes the following formulas for converting the one into the other:

$$a_x = {}_{|n}a_x + {}_{n|}a_x$$

$${}_{n|}a_x = a_x - {}_{|n}a_x$$

$${}_{|n}a_x = a_x - {}_{n|}a_x$$

DEFERRED TEMPORARY AND OTHER INTERCEPTED ANNUITIES.

An annuity, the first payment deferred more than one year and payments thereafter limited, is known as an intercepted annuity. This, however, is a general term which may also be applied to an interrupted life or temporary annuity where payments are made for a term, discontinued for a term and then permanently or temporarily resumed. A more definite and specific name for the annuity in question would therefore be a deferred temporary annuity.

Suppose, then, it were required to ascertain the value of an annuity on a life aged x to begin in n years and then continue for m years, subject to survival. Let us designate this annuity by the algebraic notation ${}_{n|}{}_{m}a_x$. Remembering, then, that an annuity is composed of a series of pure endowments, we have

$$
\begin{aligned}
{}_{n|}{}_{m}a_x &= {}_{n+1}E_x + {}_{n+2}E_x + \quad . \; . \; . \; . \; . \; . \; . \quad {}_{n+m}E_x \\
&= \frac{v^{n+1}\, l_{x+n+1} + v^{n+2}\, l_{x+n+2} + \quad . \; . \quad v^{n+m}\, l_{x+n+m}}{l_x} \\
&= \frac{D_{x+n+1} + D_{x+n+2} + \quad . \; . \; . \; . \; . \quad D_{x+n+m}}{D_x}
\end{aligned}
$$

The numerator of this fraction may easily be expressed in terms of N. The sum of the series of D above D_{x+n} by definition equals N_{x+n}; the sum of the series above D_{x+n+m} equals N_{x+n+m}. Consequently the sum of the values between D_{x+n} and $D_{x+n+m+1}$, equals $N_{x+n} - N_{x+n+m}$. So our formula takes the form

$$
{}_{n|}{}_{m}a_x = \frac{N_{x+n} - N_{x+n+m}}{D_x}
$$

The same formula may be derived from the following relations: If we deduct from the value of a life annuity deferred n years, the value of a life annuity deferred n + m years, the remainder will be the value of the intercepted annuity, deferred n years and then limited to m years. Thus

$$
\begin{aligned}
{}_{n|m}a_x &= {}_{n|}a_x - {}_{n+m|}a_x \\
&= \frac{N_{x+n}}{D_x} - \frac{N_{x+n+m}}{D_x} = \frac{N_{x+n} - N_{x+n+m}}{D_x}
\end{aligned}
$$

Suppose, now, that it were required to ascertain the value of an annuity running for n years, then interrupted for m years and then resumed for life. We may consider this as, first, a life annuity, less an annuity deferred n years and then limited to m years or, second, as a temporary annuity for n years plus a life annuity deferred n + m years. In the former aspect, the problem takes this form:

$$
\begin{aligned}
a_x - {}_{n|m}a_x &= ({}_1E_x + {}_2E_x + {}_3E_x + \;.\;.\;.\;.\;.\;.\;.\;.) \\
&\quad - ({}_{n+1}E_x + {}_{n+2}E_x + \;.\;.\;.\;.\;.\;.\;.\;.\; {}_{n+m}E_x) \\
&= \frac{(v\,l_{x+1} + v^2\,l_{x+2} + \;.\;\;.) - (v^{n+1}l_{x+n+1} + \;.\;\;.\; v^{n+n}l_{x+n+m})}{l_x} \\
&= \frac{(D_{x+1} + D_{x+2} + D_{x+3} + \;.\;\;.) - (D_{x+n+1} + D_{x+n+2} + \;.\; D_{x+n+m})}{D_x} \\
&= \frac{N_x - (N_{x+n} - N_{x+n+m})}{D_x} = \frac{N_x - N_{x+n} + N_{x+n+m}}{D_x}
\end{aligned}
$$

In the other aspect, the problem takes the form:

$$
\begin{aligned}
|_n a_x + {}_{n+m|}a_x &= ({}_1E_x + {}_2E_x + \;.\;.\;.\;.\;.\;.\;.\;.\;.\; {}_nE_x) \\
&\quad + ({}_{n+m+1}E_x + {}_{n+m+2}E_x + \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.) \\
&= \frac{(v\,l_{x+1} + v^2\,l_{x+2} + \;.\;\;.\; v^n\,l_{x+n}) + (v^{n+m+1}\,l_{x+n+m+1} + \;.\;\;.\;\;.)}{l_x} \\
&= \frac{(D_{x+1} + D_{x+2} + \;.\;\;.\; D_{x+n}) + (D_{x+n+m+1} + D_{x+n+m+2} + \;.\;\;.)}{D_x} \\
&= \frac{N_x - N_{x+n} + N_{x+n+m}}{D_x} \quad \text{as before}
\end{aligned}
$$

Suppose, again, that it were required to ascertain the value of an annuity running for n years, then interrupted for m years, then limited to m′ years more. This would be composed of a temporary annuity plus a deferred temporary annuity. Discarding the intervening steps, we may give the result in commutation symbols, thus:

$$|_{n}a_x + {}_{n+m}|_{m'}a_x = \frac{(N_x - N_{x+n}) + (N_{x+n+m} - N_{x+n+m+m'})}{D_x}$$

Suppose, again, that it were required to ascertain the value of an annuity deferred n years, then limited to m years, then interrupted for m′ years and then again limited to m″ years. This would be composed of two deferred temporary annuities and would take the form:

$$ {}_{n}|_{m}a_x + {}_{n+m+m'}|_{m''}a_x = \frac{(N_{x+n} - N_{x+n+m}) + (N_{x+n+m+m'} - N_{x+n+m+m'+m''})}{D_x}$$

In a similar manner all complex forms of intercepted annuities may be split up into temporary, deferred temporary and life annuities and their value so ascertained.

DEFINITION OF "PREMIUM."

IN developing formulas for the present values of life, temporary, deferred and intercepted annuities, we have at the same time discovered formulas, on the same assumptions, for the lowest price, paid in advance, for which a company could afford to sell such annuity contracts; since no company could afford to sell them at less than cost and, if it made a practice of selling below cost, no company could, from the proceeds of such sales, carry out its contracts. As it is commonly the business of actuaries in making these computations to discover this lowest price, they are wont to consider the resultant figures as *net premiums* instead of *present values*.

The Institute of Actuaries' Text-Book defines Premium as follows: "The sum, whether single or periodical, which is payable in consideration of a benefit, is usually called a premium." This definition does not include the idea that the premium must be sufficient or at least equal to the present value of the benefit, nor indeed does that idea properly enter into the definition of the word. The distinction is all the more important as the actuary in his ordinary work is engaged in finding present values or *sufficient* premiums and thus always has the question of the sufficiency of a premium in view. It is beyond question, however, that a sum of money paid and received in consideration for an insurance, endowment or annuity is a premium, whether it be sufficient or not from an actuarial standpoint. A very simple definition would seem to be: "A premium is the price of an insurance, endowment or annuity."

Deeming it desirable to have as authoritative a definition as possible, I have collated the following, mainly by personal correspondence and interviews:

"*Premium, Gross.*—The amount which must be paid at fixed dates named in the policy to keep the insurance in force."—*Principles and Practice of Life Insurance.*

"The price or amount paid for insurance."—*Standard Dictionary.*

"The amount paid or agreed to be paid in one sum or periodically to insurers as the consideration for a contract of insurance."—*Century Dictionary.*

"Premium in life insurance is a sum or sums stipulated to be paid to the company in consideration of future insurance benefits."—*Emory McClintock, LL. D., Actuary Mutual Life Insurance Company, and President Actuarial Society of America.*

"The price charged for insurance."—*Asa S. Wing, Actuary Provident Life and Trust Company.*

"Premium is the sum, fixed in advance, which is paid to an insuring party for insurance."—*Rufus W. Weeks, Actuary New York Life Insurance Company.*

"Premium, in insurance, is an agreed payment, made in advance, to purchase indemnity."—*D. H. Wells, Actuary Connecticut Mutual Life Insurance Company.*

"The premium, in life insurance, is the price paid or agreed to be paid as the consideration for the contract."—*J. A. De Boer, Actuary National Life Insurance Company.*

"Premium is any sum paid to a company to secure the issue of or to maintain a policy."—*Walter C. Wright, Actuary New England Mutual Life Insurance Company.*

"A consideration, payable in one sum or instalments, for a contract for a payment contingent upon life or death."—*David Parks Fackler, Consulting Actuary.*

"Premium, as used in life insurance, is the amount paid or agreed to be paid by the insured to the company as consideration money for the policy."—*B. J. Miller, Actuary Mutual Benefit Life Insurance Company.*

"A premium in insurance is the price received by an underwriter for a contract of insurance."—*Jesse J. Barker. Actuary Penn Mutual Life Insurance Company.*

"Premium is the consideration required for the benefits guaranteed."—*J. M. Craig, Actuary Metropolitan Life Insurance Company.*

"*Gross Premium.*—The amount or amounts of money which, if paid at the time or times stipulated in the contract, will, plus interest, provide for the payment of the policy's share of the company's claims and expenses."—*J. G. Van Cise, Assistant Actuary Equitable Life Assurance Society.*

"Premium in insurance is the stipulated consideration to be paid, either in one sum or periodically, under a contract securing a benefit or compensation for loss upon the happening of the contingency insured against."—*L. G. Fouse, President and Actuary Fidelity Mutual Life Association.*

W. D. Whiting, consulting actuary, endorses the definition in the Standard Dictionary; H. W. St. John, actuary Ætna Life Insurance Company, endorses the definitions by the Century Dictionary and the Institute of Actuaries' Text-Book, and C. A. Loveland, actuary of the Northwestern Mutual Life Insurance Company, adopts the Century Dictionary definition as his own.

WHOLE LIFE INSURANCE.

English and American actuaries in considering insurance have assumed that the sum insured is payable at the close of the policy or insurance year in which the insured dies. In their ordinary computations they assume that the deaths at any age are evenly distributed throughout the year and that, consequently, on the average deaths occur when the year is half over. Taking the sum payable as due at the close of the policy year is therefore assuming that the sum insured is payable on the average six months after the death of the insured.

This assumption is plainly erroneous in these days of prompt payment of death claims. French and some other European actuaries, coming later into the field than the English, discarded this fallacy. They for a long time counted the sum insured as payable at the beginning instead of the close of the policy year, considering—wisely, it seems to me—that money to be devoted to paying claims within the year will earn no interest. Subsequently, however, in the interest of scientific exactitude, they shifted their position, now assuming that the sum insured is payable at the middle of the policy year.

Though the English assumption is more at variance with the facts than either of the others, it has persisted and is so embalmed in all the text-books, tables of rates and reserves and computations that it would be rash to attempt to reconstruct the whole science to conform to more accurate assumptions, even if the congress of actuaries had not in the interest of uniformity indorsed the English notations as employed in the English text-book. Accordingly the general formulas in this book will be based on the same assumption as the English.

There is so close a connection between a life annuity and a life insurance that the value of the latter may be readily derived from the former. To define the two is to make the close relationship apparent. A life annuity of one dollar is a promise to pay one dollar at the close of each year that the annuitant survives; a life insurance of one dollar is a promise to pay one dollar at the close of the year that he fails to survive. It is therefore supplementary to a life annuity.

But let us suppose that two annuities have been issued on a life aged x, each for one dollar payable at the end of the year, but one for each year that he enters and the other for each year that he survives. The value of the latter is a_x; the value of the former one year from now, after the first payment has been collected, will be a_x or before that payment is collected, $1 + a_x$ and its present value accordingly is $1 + a_x$ discounted by interest for one year or $v\,(1 + a_x)$.

But the difference between the value of a promise to pay one dollar at the end of each year which the annuitant enters and the value of a promise to pay one dollar at the end of each year which he survives is the value of a promise to pay one dollar at the end of the year which he enters, but does not survive, or, in other words, the present value of an insurance on his life of one dollar. Let A_x be the symbol of the present value or net single premium of an insurance of $1 at age x; then we have

$$A_x = v(1 + a_x) - a_x$$

Note.—This is according to the English notation; according to the American, Π_x is the symbol for a single premium and the formula becomes

$$\Pi_x = v\,(A_x) - (A_x - 1) = 1 + v\,(A_x) - A_x$$

WHOLE LIFE INSURANCE.—COMMUTATION COLUMNS C AND M.

In the last lesson a formula was derived to give the value of an insurance of one dollar in terms of an annuity of one dollar at the same age. We may, however, get the value of an insurance from the mortality table directly, as follows:

Suppose l lives at age x are insured on the same day for one dollar each, payable at the end of the year in which death occurs. The total to be ultimately paid is then l_x dollars which will fall due as the deaths occur, that is, d_x dollars at the end of one year, d_{x+1} at the end of two years, etc., until all have died. The present values of these sums are $v\,d_x$, $v^2\,d_{x+1}$, $v^3\,d_{x+2}$, $v^4\,d_{x+3}$, etc. The aggregate present value is $v\,d_x + v^2\,d_{x+1} + v^3\,d_{x+2} + \ldots$, etc., and the value of each insurance is that aggregate divided by l_x, the number of lives, thus:

$$A_x = \frac{v\,d_x + v^2\,d_{x+1} + v^3\,d_{x+2} + \ldots\ldots\ldots}{l_x}$$

Just as we did in the case of the annuities, we may multiply numerator and denominator by v^x without altering the value. We thus get

$$A_x = \frac{v^{x+1}\,d_x + v^{x+2}\,d_{x+1} + v^{x+3}\,d_{x+2} + \ldots\ldots}{v^x\,l_x}$$

But $v^x\,l_x$ we have expressed by the symbol D_x and the numerator now consists of the sum of a series of the form $v^{x+1}d_x$, which we may derive once for all and call C_x, C_{x+1}, C_{x+2}, etc. We then have

$$A_x = \frac{C_x + C_{x+1} + C_{x+2} + \ldots\ldots\ldots\ldots}{D_x}$$

This formula is evidently general; for let us take any other age as for instance, $x + n$. We get

$$A_{x+n} = \frac{v\,d_{x+n} + v^2\,d_{x+n+1} + v^3\,d_{x+n+2} + \quad . \quad . \quad . \quad .}{l_{x+n}}$$

Multiplying numerator and denominator by v raised to a power equal to the suffix, $x + n$, we have

$$A_{x+n} = \frac{v^{x+n+1}\,d_{x+n} + v^{x+n+2}\,d_{x+n+1} + \quad . \quad . \quad . \quad . \quad .}{v^{x+n}\,l_{x+n}}$$

But this denominator is equivalent to D_{x+n} and this numerator is equal to the sum of the values of C beginning with C_{x+n}. Consequently, we save labor by summing these values for each age once for all; which has been done and the sums given the symbol M. Thus our formulas take the very simple form:

$$A_x = \frac{M_x}{D_x} \text{ and}$$

$$A_{x+n} = \frac{M_{x+n}}{D_{x+n}}$$

Except in its aggregate or summed form of M, C is so little used that it is seldom tabulated, the values of M being alone given in the commutation tables.

It is noteworthy that $D_x = v^x\,l_x$ and $C_x = v^{x+1}\,d_x$; substituting for d_x, its value in terms of l, we have

$$C_x = v^{x+1}\,(l_x - l_{x+1}) = v\,D_x - D_{x+1}$$

Since N_x = a series of values of D_{x+1} or $\Sigma\,D_{x+1}$,
N_{x-1} = a series of values of D_x or $\Sigma\,D_x$ and
M_x = a series of values of C_x or $\Sigma\,C_x$, it follows
that: $M_x = v\,N_{x-1} - N_x$

So the value of A_x may be derived from the N and D columns directly, thus:

$$A_x = \frac{M_x}{D_x} = \frac{v\,N_{x-1} - N_x}{D_x}$$

Note.—The American symbol for a net single premium for whole life insurance is Π instead of A; but unlike the values of N, the values of M exactly correspond in the English and American notation. In American notation, the formulas take the form

$$\Pi_x = \frac{M_x}{D_x} = \frac{v\,N_x - N_{x+1}}{D_x}$$

DEFERRED, TEMPORARY AND DEFERRED TEMPORARY INSURANCE.

If an insurance on a life now aged x is not to begin until a later age, it is known as a deferred insurance. Suppose it is to begin in n years; then, after n years has elapsed and the life is aged x + n, the value is

$$A_{x+n} = \frac{M_{x+n}}{D_{x+n}}.$$

Its present value, therefore, would be, adopting ${}_{n|}A_x$ as the symbol for that value:

$${}_{n|}A_x = {}_nE_x \frac{M_{x+n}}{D_{x+n}} = \frac{D_{x+n}}{D_x} \times \frac{M_{x+n}}{D_{x+n}} = \frac{M_{x+n}}{D_x}.$$

If an insurance is to extend only for a limited term, it is called a temporary insurance and its value is designated as $|_n A_x$. The value of a temporary insurance for one year from age x is the present value of the aggregate number of dollars, d_x, payable at the end of one year to the heirs of those of l_x insured who do not survive, thus

$|_1 A_x = \frac{v\,d_x}{l_x}$ and, multiplying numerator and denominator by v^x,

$$|_1 A_x = \frac{v^{x+1} d_x}{v^x l_x} = \frac{C_x}{D_x}.$$

The value also of one dollar payable at the end of two years to each who survives one year but not the second is $\frac{v^2 d_{x+1}}{l_x}$, which being similarly modified becomes $\frac{v^{x+2} d_{x+1}}{v^x l_x}$ or $\frac{C_{x+1}}{D_x}$.

The value, then, of an insurance for both years will be the sum of these and generally

$$|_n A_x = \frac{C_x + C_{x+1} + \ldots\ldots\ldots C_{x+n-1}}{D_x}$$

But the numerator of this fraction is M_x or the sum of the values of C beginning with C_x less M_{x+n} or the sum of the values of C beginning with C_{x+n} which gives us the formula

$$|_n A_x = \frac{M_x - M_{x+n}}{D_x}.$$

An insurance beginning at some time in the future and then running for a limited time is known as a deferred temporary or intercepted insurance, and its value is given the symbol ${}_n|_m A_x$, which means an insurance which begins in n years and then continues for m years, or from age x + n to age x + n + m.

We have just found that the value of a dollar payable at the end of the second year, if one survive the first year and fail to survive the second, is

$$\frac{v^2 d_{x+1}}{l_x} = \frac{v^{x+2} d\, l_{x+1}}{v^x l_x} = \frac{C_{x+1}}{D_x}.$$ This is the value of an insurance for one year, deferred one year and the formula may be made general, thus:

$${}_n|_1 A_x = \frac{C_{x+n}}{D_x}.$$

A temporary insurance deferred n years and then continuing for m years would be composed of all these insurances for one year from x + n up to x + n + m; hence

$${}_n|_m A_x = \frac{C_{x+n} + C_{x+n+1} + \ldots\ldots C_{x+n+m-1}}{D_x}.$$

But the sum of these values of C is equal to M_{x+n} or the sum of the values of C from age x + n less M_{x+n+m}, or the sum of the values of C from age x + n + m. From which we derive:

$$ {}_{n|m}A_x = \frac{M_{x+n} - M_{x+n+m}}{D_x}. $$

From the interrelations of these various insurances we get the following conversion formulas:

$$ {}_{n|}A_x = A_x - {}_{|n}A_x = {}_nE_x\, A_{x+n} $$
$$ {}_{|n}A_x = A_x - {}_{n|}A_x = A_x - {}_nE_x\, A_{x+n} $$
$$ {}_{n|m}A_x = {}_{n|}A_x - {}_{n+m|}A_x = {}_nE_x\, {}_{|m}A_{x+n} $$

Note.—The American notation is as follows:

Deferred insurance ${}_n\Pi_x$
Temporary insurance Π_{xn}
Deferred temporary insurance ${}_{n|m}\Pi_x$

CONVERSION FORMULAS—ANNUITIES AND INSURANCES.

EARLY in our consideration of pure endowments and annuities we learned that

${}_nE_x = v^n\ {}_np_x$ and that

$a_x = v\,p_x + v^2\ {}_2p_x + v^3\ {}_3p_x + \ldots\ldots\ldots$

The last formula may be stated also as follows:

$a_x = \Sigma\, v^n\ {}_np_x$

The principle is that the value of a sum, payment being contingent, is the value of a like sum certain, multiplied by the probability of its becoming due. Applying this to insurance, we have for an insurance of one year:

$|_1A_x = v\,q_x$

The value of a second year's insurance becomes:

${}_1|_1A_x = v^2\ {}_1|q_x$

The value of the two years' insurance is, then:

$|_2A_x = v\,q_x + v^2\ {}_1|q_x$ or generally:

$|_nA_x = v\,q_x + v^2\ {}_1|q_x + \ldots\ldots\ v^n\ {}_{n-1}|q_x$

This makes the value of a whole life insurance

$A_x = v\,q_x + v^2\ {}_1|q_x + \ldots = \Sigma\, v^n\ {}_{n-1}|q_x$

But the probability of dying in any year is equal to the probability of surviving to the beginning of that year, less the probability of surviving to its close; thus:

${}_{n-1}|q_x = {}_{n-1}p_x - {}_np_x$.

Substituting, we get:

$A_x = \Sigma\, v^n\,({}_{n-1}p_x - {}_np_x)$.

We have already found that the value of one dollar at the close of the year which a life fails to survive is the difference between the value of an annuity payable at the end of each year the life begins and the value of an annuity payable at the end of each year the life survives, thus:

$A_x = v\,(1 + a_x) - a_x$.

We may now trace further correspondences and interrelations as follows:

First, suppose one dollar will produce i interest each year for a life aged x, together with a return of the dollar with one year's interest $(1+i)$, at the end of the year the life does not survive. The value of the interest during the life is that of a life annuity for that amount or $i\,a_x$. He has paid down one dollar in all, and it follows that the remainder, after deducting the value of the annuity, is the value of the $1+i$ payable at the end of the year the life fails, thus: an insurance of $1+i$ costs $1-i\,a_x$. From this we derive the value of an insurance of one dollar, thus:

$$A_x = \frac{1 - i\,a_x}{1+i}$$

Second; if one dollar certain were due in one year, the value would be v. But, not being due until the end of the year in which the life fails, the value is less than v by an amount equal to the value of the interest on v during the life or $v\,i\,a_x$; from which we get:

$A_x = v - v\,i\,a_x = v\,(1 - i\,a_x)$; or, since $v\,i$ is equivalent to the discount on one dollar certain due at the end of the year or, in other words, to the difference between one dollar and v we may also derive:

$$A_x = v - v\,i\,a_x = v - (1-v)\,a_x = v - d\,a_x.$$

Third; if the dollar were payable now, its value would be 1; not being payable until the end of the year which the life does not survive, its value is less than 1 by the whole value of the interest. The value of the interest for one year is $v\,i = d$; the value, then, of an annuity of d, payable at the close of each year upon which the life enters is $d\,(1 + a_x)$; and we get

$$A_x = 1 - d\,(1 + a_x).$$

Fourth; let a_∞ be the value of a perpetuity of one dollar at the end of each year. The value of what remains after the life fails will be $a_\infty - a_x$; but at the end of the year the life fails, the value will be one dollar due and a perpetuity of one dollar at the end of each year or $1 + a_\infty$. It follows that the value of an insurance of $1 + a_\infty$ is $a_\infty - a_x$, from which we get the value of an insurance of 1, namely:

$$A_x = \frac{a_\infty - a_x}{1 + a_\infty}.$$

These equations also yield the following values for a_x in terms of A_x:

From $A_x = v\,(1 + a_x) - a_x$,

$$A_x = v + v\,a_x - a_x = v + (v - 1)\,a_x = v - (1 - v)\,a_x$$

$$a_x = \frac{v - A_x}{1 - v} = \frac{v - A_x}{d}$$

From $A_x = \frac{1 - i\,a_x}{1 + i}$,

$$(1+i)\,A_x = 1 - i\,a_x$$

$$i\,a_x = 1 - (1+i)\,A_x$$

$$a_x = \frac{1 - (1+i)\,A_x}{i}$$

From $A_x = v\,(1 - i\,a_x)$, we get also:

$$a_x = \frac{v - A_x}{d}.$$

This may also be derived from $A_x = v - (1 - v)\,a_x$.

From $A_x = 1 - d\,(1 + a_x) = 1 - d - d\,a_x$.

$$d\,a_x = 1 - d - A_x$$

$$a_x = \frac{1 - d - A_x}{d} = \frac{1 - A_x}{d} - 1.$$

From $A_x = \frac{a_\infty - a_x}{1 + a_\infty}$

$$(1 + a_\infty) A_x = a_\infty - a_x$$

$$a_x = a_\infty - (1 + a_\infty) A_x$$

Referring again to the fundamental formula, $A_x = v(1 + a_x) - a_x$ and the explanation that the value of an insurance of one dollar at the end of the year not survived is the difference between the value of an annuity of one dollar at the end of each year the life enters and of a like annuity at the end of each year the life survives, it is at once clear that the same reasoning applies to temporary insurance and annuities, so that we may substitute as follows:

$$|_n A_x = v(1 + |_{n-1}a_x) - |_n a_x.$$

This expression will also answer for the single premium for an endowment insurance, payable in n years or at prior death, if it is suitably modified. In such case, not so much is to be deducted, by the pure endowment ${}_nE_x$ which is the last quantity in the series $E_x + {}_2E_x + \dots + {}_nE^x = |_n a_x$: that is to say, $|_{n-1}a_x$ is deducted instead, giving:

$$|_n \text{Æ}_x = v(1 + |_{n-1}a_x) - |_{n-1}a_x$$

$$= 1 - d(1 + |_{n-1}a_x)$$

ENDOWMENT INSURANCES.

The pure endowment or promise to pay only on the condition of survival is not a form of contract commonly purchased; but a contract promising to pay a sum in event either of death during a term or survival of that term, known as endowment insurance, is common.

The value of an endowment insurance of one dollar on a life aged x and due in n years or at prior death, we may designate as $|_n Æ_x$. It is composed of the successive values of one year's insurance plus the value of a pure endowment due in n years, thus:

$$|_n Æ_x = v\, q_x + v^2 \;{}_1|q_x + \;.\;\;.\;\; v^n \;{}_{n-1}|q_x + {}_nE_x\,.$$
$$= |_n A_x + {}_nE_x\,.$$

In other words, by its very definition the endowment insurance is composed of a temporary insurance for n years and a pure endowment in n years.

We have found the value of $|_n A_x$ expressed in commutation symbols to be $\frac{M_x - M_{x+n}}{D_x}$ and the value of ${}_nE_x$, $\frac{D_{x+n}}{D_x}$.

Combining these, then, we have:

$$|_n Æ_x = \frac{M_x - M_{x+n}}{D_x} + \frac{D_{x+n}}{D_x} = \frac{M_x - M_{x+n} + D_{x+n}}{D_x}$$

Referring to the conversion formula, $A_x = \frac{1 - i\, a_x}{1 + i}$ and its explanation, which was: A dollar down produces i interest at the end of each year the life survives and produces 1 + i, principal and interest, at the end of the year the life does not survive; wherefore an insurance of 1 + i is valued at 1 less the value of the interest during survival or i a_x and an insurance of 1 is valued at $\frac{1 - i\, a_x\,.}{1 + i}$

In the foregoing, the English notation has been used throughout. The Institute of Actuaries' text-book, however, also adopts the following alternative symbols: $A_{x\overline{n}|}$ for $|_n Æ_x$ and $a_{x\overline{x}|}$ for $|_n a_x$. It introduces these after the others have been made familiar, but expresses a preference for them. With these changes the conversion formula becomes:

n/

$$A_{x\overline{n}|} = v(1 + a_{x\overline{n-1}|}) - a_{x\overline{n-1}|}$$
$$= 1 - d(1 + a_{x\overline{n-1}|})$$

The American symbol for a single premium for endowment insurance is ${}_{e}\Pi_{xn}$. The same commutation formulas are applicable.

ENDOWMENTS AND INSURANCE WITH RETURN OF NET SINGLE PREMIUM.

SUPPOSE a pure endowment due in n years with return of net single premium if death occurs in n years. The value is composed of the value of an endowment of 1 in n years and an insurance for n years of the net single premium, which we will designate by E (slanting, instead of vertical), thus:

$$E = {}_{n}E_{x} + E\,|_{n}A_{x} = \frac{D_{x+n} + E\,(M_{x} - M_{x+n})}{D_{x}}$$

$$D_{x}\,E = D_{x+n} + E\,(M_{x} - M_{x+n})$$

$$E\,[D_{x} - (M_{x} - M_{x+n})] = D_{x+n}$$

$$E = \frac{D_{x+n}}{D_{x} - (M_{x} - M_{x+n})}.$$

Suppose an insurance for n years, with return of net single premium. We have now an insurance of 1 for n years and an insurance for n years of the net single premium, which we will designate by the symbol A (slanting, not vertical), which symbol is used in the English notation for *all special or unusual net single premiums for insurance.* Thus we get:

$$A = (1 + A)\,|_{n}A_{x} = |_{n}A_{x} + (|_{n}A_{x})\,A$$

$$A\,(1 - |_{n}A) = |_{n}A_{x}$$

$$A = \frac{|_{n}A_{x}}{1 - |_{n}A_{x}}$$

Or in commutation symbols, we get:

$$A = (1 + A)\,|_{n}A_{x} = \frac{(1 + A)\,(M_{x} - M_{x+n})}{D_{x}}$$

$$D_{x}\,A = (1 + A)\,(M_{x} - M_{x+n})$$

$$A\,[D_{x} - (M_{x} - M_{x+n})] = M_{x} - M_{x+n}$$

$$A_{x} = \frac{M_{x} - M_{x+n}}{D_{x} - (M_{x} - M_{x+n})}.$$

Suppose a whole life insurance, with return of net single premium; its value is composed of an insurance of 1 plus an insurance of A, thus:

$$A = (1 + A)\ \mathrm{A}_x = \mathrm{A}_x + (\mathrm{A}_x)A$$
$$A\ (1 - \mathrm{A}_x) = \mathrm{A}_x$$
$$A = \frac{\mathrm{A}_x}{1 - \mathrm{A}_x}$$

Or in commutation symbols, we get:

$$A = (1 + A)\ \mathrm{A}_x = \frac{(1 + A)\ \mathrm{M}_x}{\mathrm{D}_x}$$
$$\mathrm{D}_x A = (1 + A)\ \mathrm{M}_x$$
$$A\ (\mathrm{D}_x - \mathrm{M}_x) = \mathrm{M}_x$$
$$A = \frac{\mathrm{M}_x}{\mathrm{D}_x - \mathrm{M}_x}.$$

Suppose an endowment insurance with return of premium, either at death or upon survival; the value is composed of a pure endowment in n years of 1 + A plus an insurance for n years of 1 + A, thus:

$$A = (1 + A)\ {}_{n}\mathrm{E}_x + (1 + A)\ |_{n}\mathrm{A}_x$$
$$= \frac{(1 + A)\ \mathrm{D}_{x+n} + (1 + \mathrm{A})\ (\mathrm{M}_x - \mathrm{M}_{x+n})}{\mathrm{D}_x}$$
$$= \frac{(1 + A)\ (\mathrm{D}_{x+n} + \mathrm{M}_x - \mathrm{M}_{x+n})}{\mathrm{D}_x}$$
$$\mathrm{D}_x A = \mathrm{D}_{x+n} + \mathrm{M}_x - \mathrm{M}_{x+n} + A\ (\mathrm{D}_{x+n} + \mathrm{M}_x - \mathrm{M}_{x+n})$$
$$A\ [\mathrm{D}_x - (\mathrm{D}_{x+n} + \mathrm{M}_x - \mathrm{M}_{x+n})] = \mathrm{D}_{x+n} + \mathrm{M}_x - \mathrm{M}_{x+n}$$
$$A = \frac{\mathrm{D}_{x+n} + \mathrm{M}_x - \mathrm{M}_{x+n}}{\mathrm{D}_x - (\mathrm{D}_{x+n} + \mathrm{M}_x - \mathrm{M}_{x+n})}$$

Suppose a like case, except that the premium is only returnable if death occurs in n years; the value is composed of an endowment of 1 in n years and an insurance of 1 + A for n years, thus:

$$A = {}_{n}\mathrm{E}_x + (1 + A)\ {}^{1}_{n}\mathrm{A}_x = \frac{\mathrm{D}_{x+n} + (1 + A)\ (\mathrm{M}_x - \mathrm{M}_{x+n})}{\mathrm{D}_x}$$

$$\mathrm{D}_x\, A = \mathrm{D}_{x+n} + \mathrm{M}_x - \mathrm{M}_{x+n} + A\,(\mathrm{M}_x - \mathrm{M}_{x+n})$$

$$A\,[\mathrm{D}_x - (\mathrm{M}_x - \mathrm{M}_{x+n})\,] = \mathrm{D}_{x+n} + \mathrm{M}_x - \mathrm{M}_{x+n}$$

$$A = \frac{\mathrm{D}_{x+n} + \mathrm{M}_x - \mathrm{M}_{x+n}}{\mathrm{D}_x - (\mathrm{M}_x - \mathrm{M}_{x+n})}$$

Again, suppose a like case, except that the premium is returnable only with the endowment upon survival; the value is composed of an endowment in n years of $1 + A$ and an insurance of 1 for n years, thus:

$$A = (1 + A)\,{}_n\mathrm{E}_x + \vert_n\mathrm{A}_x = \frac{(1 + A)\,\mathrm{D}_{x+n} + \mathrm{M}_x - \mathrm{M}_{x+n}}{\mathrm{D}_x}$$

$$\mathrm{D}_x\, A = (1 + A)\,\mathrm{D}_{x+n} + \mathrm{M}_x - \mathrm{M}_{x+n}$$

$$A\,(\mathrm{D}_x - \mathrm{D}_{x+n}) = \mathrm{D}_{x+n} + \mathrm{M}_x - \mathrm{M}_{x+n}$$

$$A = \frac{\mathrm{D}_{x+n} + \mathrm{M}_x - \mathrm{M}_{x+n}}{\mathrm{D}_x - \mathrm{D}_{x+n}}$$

It should be observed that these formulas cover only the cases of policies with return of *single* premiums. Policies with return of annual premiums will be dealt with after we have considered the general problems of annual premiums. An insurance with return of a single premium is not equivalent in value to an insurance with return of the corresponding annual premium.

Note.—A general formula for the single premium of a return-premium insurance, the premium without premium-return being known, may be found thus: The full amount insured is $1 + A$; but A, the known premium for an insurance of 1, bears the same ratio to 1 that A bears to $1 + A$, that is:

$$1 : \mathrm{A} :: 1 + A : A$$

$$A = \mathrm{A}\,(1 + A)$$

$$A\,(1 - \mathrm{A}) = \mathrm{A}$$

$$A = \frac{\mathrm{A}}{1 - \mathrm{A}}$$

ENDOWMENTS AND INSURANCES WITH RETURN OF GROSS SINGLE PREMIUMS.

IT is customary to increase the net premium by a margin or loading to make the gross or office premium, actually charged for the benefit. This loading may take the form of a percentage addition to the net premium or of the arbitrary addition of a lump sum, or both. That a premium is gross instead of net is indicated by the "prime", thus: A'_x, ${}_nE'_x$, etc. The relations between gross and net premiums in each of the foregoing cases, respectively, indicating a percentage addition by k and a lump addition by c, are as follows:

$A' = (1 + k)\,A$ . . . . percentage only
$A' = A + c$ . . . . . constant only.
$A' = (1 + k)\,A + c$ . . . percentage and constant

To avoid repetition, we will employ the last of these; because formulas drawn from it will fit each of the other cases by substituting zero for c in the first instance and for k in the second instance.

Suppose, then, a pure endowment in n years with the gross premium, E', returnable if death occur in n years; the value is composed of an endowment of 1 in n years and an insurance of E' for n years, thus:

$$E = {}_nE_x + E' \,|\, {}_nA_x = \frac{D_{x+n} + E'\,(M_x - M_{x+n})}{D_x}$$

$$= \frac{D_{x+n} + [\,(1 + k)\,E + c]\,(M_x - M_{x+n})}{D_x}$$

$$D_x\,E = D_{x+n} + c\,(M_x - M_{x+n}) + [\,(1 + k)\,E]\,(M_x - M_{x+n})$$

$$E\,[D_x - (1 + k)\,(M_x - M_{x+n})\,] = D_{x+n} + c\,(M_x - M_{x+n})$$

$$E = \frac{D_{x+n} + c\,(M_x - M_{x+n})}{D_x - (1 + k)\,(M_x - M_{x+n})}$$

Suppose an insurance for n years with return of gross premium; this is an insurance of 1 plus an insurance of A', thus:

$$A = (1 + A')\,|_{n}A_x = \frac{(1 + A')(M_x - M_{x+n})}{D_x}$$

$$= \frac{[1 + (1 + k)A + c](M_x - M_{x+n})}{D_x}$$

$$D_x A = (1 + c)(M_x - M_{x+n}) \div [(1 + k)A](M_x - M_{x+n})$$

$$A[D_x - (1 + k)(M_x - M_{x+n})] = (1 + c)(M_x - M_{x+n})$$

$$A = \frac{(1 + c)(M_x - M_{x+n})}{D_x - (1 + k)(M_x - M_{x+n})}$$

Suppose an insurance for life with return of gross premium; it becomes:

$$A = (1 + A')A_x = [(1 + (1 + k)A + c]A_x$$

$$A[1 - (1 + k)A_x] = (1 + c)A_x$$

$$A = \frac{(1 + c)A_x}{1 - (1 + k)A_x}$$

In commutation symbols, we get:

$$A = \frac{(1 + A')M_x}{D_x} = \frac{[1 + (1 + k)A + c]M_x}{D_x}$$

$$D_x A = (1 + c)M_x + [(1 + k)A]M_x$$

$$A[D_x - (1 + k)M_x] = (1 + c)M_x$$

$$A = \frac{(1 + c)M_x}{D_x - (1 + k)M_x}$$

In the case of an endowment with return of gross premium, either at death or upon survival, we have a temporary insurance of $1 + A'$ and a pure endowment of $1 + A'$, worth respectively: $\frac{(M_x - M_{x+n})(1 + A')}{D_x}$ and $\frac{D_{x+n}(1 + A')}{D_x}$; and combined:

$$A = \frac{(M_x - M_{x+n} + D_{x+n})(1 + A')}{D_x}$$

$$= \frac{(M_x - M_{x+n} + D_{x+n})[1 + (1 + k)A + c]}{D_x}$$

$$D_x A = (M_x - M_{x+n} + D_{x+n})\,[1 + (1 + k)\,A + c]$$
$$= (M_x - M_{x+n} + D_{x+n})\,(1 + c) + (M_x - M_{x+n} + D_{x+n})\,[\,(1 + k)\,A]$$
$$A\,[D_x - (1 + k)\,(M_x - M_{x+n} + D_{x+n})\,] = (1 + c)\,(M_x - M_{x+n} + D_{x+n})$$
$$A = \frac{(1 + c)\,(M_x - M_{x+n} + D_{x+n})}{D_x - (1 + k)\,(M_x - M_{x+n} + D_{x+n})}$$

In an endowment policy returning gross premiums only in event of death, we have an endowment of 1 and an insurance of $1 + A'$ and combined, in commutation symbols:

$$A = \frac{(M_x - M_{x+n})\,(1 + A') + D_{x+n}}{D_x}$$
$$= \frac{(M_x - M_{x+n})\,[1 + (1 + k)\,A + c] + D_{x+n}}{D_x}$$
$$D_x A = (M_x - M_{x+n})\,(1 + c) + (M_x - M_{x+n})\,[\,(1+k)\,A] + D_{x+n}$$
$$A\,[D_x - (1 + k)\,(M_x - M_{x+n})\,] = (1 + c)\,(M_x - M_{x+n}) + D_{x+n}$$
$$A = \frac{(1 + c)\,(M_x - M_{x+n}) + D_{x+n}}{D_x - (1 + k)\,(M_x - M_{x+n})}$$

In an endowment policy returning the gross premium only in case of survival, the value is composed of an endowment of $1 + A'$ and an insurance of 1; which, combined, become, in commutation symbols:

$$A = \frac{M_x - M_{x+n} + (1 + A')\,D_{x+n}}{D_x}$$
$$= \frac{M_x - M_{x+n} + [1 + (1 + k)\,A + c]\,D_{x+n}}{D_x}$$
$$D_x A = M_x - M_{x+n} + (1 + c)\,D_{x+n} + [\,(1 + k)\,A]\,D_{x+n}$$
$$A\,[D_x - (1 + k)\,D_{x+n}] = M_x - M_{x+n} + (1 + c)\,D_{x+n}$$
$$A = \frac{M_x - M_{x+n} + (1 + c)\,D_{x+n}}{D_x - (1 + k)\,D_{x+n}}$$

It must not be forgotten that these are not formulas for single premiums, corresponding to net annual premiums to return gross annual premiums.

VARYING BENEFITS IN ANNUITIES AND INSURANCES.

We have so far developed the following commutation columns:

$$D_x = v^x l_x$$

$$N_x = D_{x+1} + D_{x+2} + \quad \ldots\ldots \quad = \Sigma D_{x+1}$$

$$C_x = v^{x+1} d_x = v^{x+1} (l_x - l_{x+1})$$

$$M_x = C_x + C_{x+1} + \quad \ldots\ldots \quad = \Sigma C_x$$

We have also developed, among others, the following commutation formulas:

$$a_x = \frac{N_x}{D_x}$$

$|_n a_x$ or $a_{x\overline{n|}} = \frac{N_x - N_{x+n}}{D_x}$, which we condense to $\frac{N_{x\overline{n|}}}{D_x}$

$$_n|a_x = \frac{N_{x+n}}{D_x}$$

$$A_x = \frac{M_x}{D_x}$$

$|_n A_x = \frac{M_x - M_{x+n}}{D_x}$, condensed to $\frac{M_{x\overline{n|}}}{D_x}$

$$_n|A_x = \frac{M_{x+n}}{D_x}$$

$|_n Æ_x$ or $A_{x\overline{n|}} = \frac{M_x - M_{x+n} + D_{x+n}}{D_x} = \frac{M_{x\overline{n|}} + D_{x+n}}{D_x}$

We have also discovered the following reciprocal relationships between the values of D and C and N and M:

$$C_x = v D_x - D_{x+1}$$

$$M_x = v N_{x-1} - N_x$$

Suppose a life annuity beginning at one sum (k) and annually increasing or decreasing by another sum (h), and designate its value by the symbol $(v\ a)_x$. The value of this annuity will be made up of the values of a number of successive annuities, thus: first, an annuity of k; second, an annuity of h, deferred one year; third, an annuity of h, deferred two years, etc. This combined value may be expressed thus:

$$(v\,a)_x = k\,a_x \pm h\,({}_1|a_x + {}_2|a_x + {}_3|a_x + \;.\;.\;.\;)$$

But the value of $k\,a_x$ is $\frac{k\,N_x}{D_x}$; the value of $h\,{}_1|a_x$ is $\frac{h\,N_{x+1}}{D_x}$; of $h\,{}_2|a_x$, $\frac{h\,N_{x+2}}{D_x}$, etc. Substituting, we get:

$$(v\,a)_x = \frac{k\,N_x \pm h\,(N_{x+1} + N_{x+2} + \;.\;.\;.\;.\;)}{D_x}$$

For convenience in solving such problems, we now create a new commutation column, containing values of N summed from each age to the end of the table and call these values S; thus:

$$\begin{aligned} S_x &= N_x + N_{x+1} + N_{x+2} + \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;. \\ &= (D_{x+1} + D_{x+2}\;.\;.\;) + (D_{x+2} + D_{x+3}\;.\;.\;.\;.\;) + \;.\;. \\ &= D_{x+1} + 2\,D_{x+2} + 3\,D_{x+3} + \;.\;.\;.\;.\;.\;.\;.\;.\;.\;. \end{aligned}$$

Substituting in the formula, we get:

$$(v\,a)_x = \frac{k\,N_x \pm h\,S_{x+1}}{D_x}$$

The case of a decreasing annuity is an infrequent, though not an impossible one in insurance mathematics. The most frequent form of a varying annuity benefit is one increasing annually by an amount equal to the original annuity. The solution of such a case becomes yet simpler; for assuming k and h in any case to be alike equal to 1, and adopting the symbol $(I\,a)_x$ to express its value, we have:

$$(I\,a)_x = \frac{N_x + S_{x+1}}{D_x} = \frac{S_x}{D_x}$$

Turning now to insurances, suppose an insurance of k to increase or decrease by h each year and designate its value by $(v\ A)_x$. The value of this insurance will be equal to the sum of an insurance of k, of an insurance of h deferred one year, another of h deferred two years, etc., thus:

$$(v\ A)_x = k\ A_x \pm h\ ({}_1|A_x + {}_2|A_x + {}_3|A_x + \ .\ .\ .\ )$$

But the value of $k\ A_x$ is $\frac{k\ M_x}{D_x}$; of $h\ {}_1|A_x$, $\frac{h\ M_{x+1}}{D_x}$, etc.

Substituting, we get:

$$(v\ A)_x = \frac{k\ M_x \pm h\ (M_{x+1} + M_{x+2} + \ .\ .\ .\ .\ )}{D_x}$$

For convenience in solving these problems, we now make yet a new commutation column by summing the values of M from each age, to the end of the table which values we call R. thus:

$$R_x = M_x + M_{x+1} + M_{x+2} + \ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .$$
$$= (C_x + C_{x+1}\ .\ .\ \ ) + (C_{x+1} + C_{x+2} + \ .\ .\ \ ) + \ .\ .$$
$$= C_x + 2\ C_{x+1} + 3\ C_{y+2} + \ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .$$

Substituting in the formula, we get:

$$(v\ A)_x = \frac{k\ M_x \pm h\ R_{x+1}}{D_x}$$

The case of a decreasing insurance is also infrequent, though not impossible; the case of an insurance increasing annually by its own amount is also infrequent, except in the form of return of premiums. But, expressing it by the symbol $(I\ A)_x$, meaning an insurance of 1, annually increased by 1, we derive the following formula from the last:

$$(I\ A)_x = \frac{M_x + R_{x+1}}{D_x} = \frac{R_x}{D_x}$$

We have seen that:

$$C_x = v\ D_x - D_{x+1}$$
$$M_x = v\ N_{x-1} - N_x$$

Since R_x is the sum of values of M beginning with M_x,

S_{x-1} of values of N beginning with N_{x-1} and S_x of the values of N, beginning with N_x, we may derive:

$$R_x = v\, S_{x-1} - S_x\,.$$

Substituting, we get:

$$(I\,A)_x = \frac{R_x}{D_x} = \frac{v\, S_{x-1} - S_x}{D_x}$$

But $S_{x-1} - S_x = N_{x-1}$ and $v\, S_{x-1} - S_x$ is less than that amount by just $d\, S_{x-1}$; substituting again:

$$(I\,A)_x = \frac{N_{x-1} - d\, S_{x-1}}{D_x}$$

But $\frac{S_x}{D_x}$ is the value of $(I\,a)_x$ by formula and $\frac{S_{x-1}}{D_x}$ of $(I\,\mathbf{a})_x$; so that we may derive the formula:

$$\begin{aligned}(I\,A)_x &= \frac{v\, S_{x-1} - S_x}{D_x}\\ &= v\,(I\,\mathbf{a})_x - (I\,a)_x\\ &= v\,[1 + a_x + (I\,a)_x] - (I\,a)_x\end{aligned}$$

TEMPORARILY VARYING BENEFITS IN ANNUITIES AND INSURANCES.

SUPPOSE an annuity of k at the beginning, increasing or diminishing yearly by h for n—1 years and then remaining constant. Let its value be designated by the symbol $(v\,\overline{n}|\,a)_x$. This value will be composed of an annuity of k, plus or minus an annuity of h deferred one year, an annuity of h deferred two years, etc., up to an annuity of h deferred n—1 years, thus:

$$(v\,\overline{n}|\,a)_x = k\,a_x \pm h\,({}_1|a_x + {}_2|a_x + \;.\;\;.\;\; {}_{n-1}|a_x)$$

But the value of $k\,a_x$ is $\frac{k\,N_x}{D_x}$; the value of $h_1\,|a_x$ is $\frac{h\,N_{x+1}}{D_x}$; of $h_2\,|a_x$, $\frac{h\,N_{x+2}}{D_x}$, etc., until $h_{n-1}|a_x = \frac{h\,N_{x+n-1}}{D_x}$ Substituting, the formula becomes:

$$(v\,\overline{n}|a)_x = \frac{k\,N_x \pm h\,(N_{x+1} + N_{x+2} + \;\;.\;\;.\;\;.\;\; N_{x+n-1})}{D_x}$$

Remembering that S_x is the sum of all values of N beginning with N_x, we at once see that the above series of values of N are equal to S_{x+1}, less the values beginning with N_{x+n} or, in other words, less S_{x+n}. Restating the formula with this substitution, we obtain:

$$(v\,\overline{n}|\,a)_x = \frac{k\,N_x \pm h\,(S_{x+1} - S_{x+n})}{D_x}$$

When k and h are each unity, adopting $(I\,\overline{n}|\,a)_x$ as the symbol of this value, the formula becomes in the case of an increasing annuity:

$$(I\,\overline{n}|\,a)_x = \frac{N_x + S_{x+1} - S_{x+n}}{D_x} = \frac{S_x - S_{x+n}}{D_x} = \frac{S_{x\,\overline{n}|}}{D_x}$$

Likewise an insurance of k increasing or decreasing by h annually for n years and then constant, represented by the symbol $(v\,\overline{n}|\,A)_x$, is composed of an insurance of k, an insurance of h deferred one year, an insurance of h deferred two years, etc., to an insurance of h deferred n—1 years, making n—1 increments, thus:

$$(v\,\overline{n}|\,A)_x = k\,A_x \pm h\,(\,{}_1|A_x + {}_2|A_x + \,.\;\;.\;\;.\,{}_{n-1}|A_x\,)$$

Substituting commutation values, we get:

$$(v\,\overline{n}|\,A)_x = \frac{k\,M_x \pm h\,(M_{x+1} + M_{x+2} + \;.\;\;.\;\;M_{x+n-1})}{D_x}$$

This series of M is plainly equivalent to R_{x+1}, or the values beginning with M_{x+1}, less R_{x+n} or the values beginning with M_{x+n}. Substituting, we have:

$$(v\,\overline{n}|\,A)_x = \frac{k\,M_x \pm h\,(R_{x+1} - R_{x+n})}{D_x}$$

In the case of an insurance of 1, increasing annually by 1 for n years and then constant, adopting the symbol $(I\,\overline{n}|A)_x$ to represent the value, this formula takes the form:

$$(I\,\overline{n}|\,A)_x = \frac{M_x + R_{x+1} - R_{x+n}}{D_x} = \frac{R_x - R_{x+n}}{D_x} = \frac{R_{x\,\overline{n}|}}{D_x}$$

Suppose, however, that the annuities and insurances are temporary; that is, suppose an annuity of k, increasing h yearly for n—1 years and then expiring. The value of this will be just as much less than $(v\,\overline{n}|a)_x$ as is equal to the value of the annuity (k + (n—1) h) to be paid after n years, which is $\frac{[k + (n - 1)\,h]\,N_{x+n}}{D_x}$. Giving the value of the temporary annuity the symbol $(v\,a)_{x\,\overline{n}|}$, we have:

$$(v\,a)_{x\,\overline{n}|} = \frac{k\,N_x \pm h\,(S_{x+1} - S_{x+n}) - [k + (n - 1)\,h]\,N_{x+n}}{D_x}$$

$$= \frac{k\,(N_x - N_{x+n}) \pm h\,[S_{x+1} - S_{x+n} - (n - 1)\,N_{x+n}]}{D_x}$$

$$= \frac{k\,(N_x - N_{x+n}) \pm h\,(S_{x+1} - S_{x+n} + N_{x+n} - n\,N_{x+n})}{D_x}$$

$$= \frac{k\,(N_x - N_{x+n}) \pm h\,(S_{x+1} - S_{x+n+1} - n\,N_{x+n})}{D_x}$$

When an annuity of 1, increasing by 1 each year for n years, and then expiring, is in question, adopting for it the symbol $(I\,a)_{x\overline{n|}}$, we have:

$$(I\,a)_{x\overline{n|}} = \frac{N_x - N_{x+n} + S_{x+1} - S_{x+n+1} - n\,N_{x+n}}{D_x}$$

$$= \frac{S_x - S_{x+n} - n\,N_{x+n}}{D_x}$$

The same course of reasoning, identically, in the case of insurances, gives us

$$(v\,A)^1_{x\overline{n|}} = \frac{k\,(M_x - M_{x+n}) \pm h\,(R_{x+1} - R_{x+n+1} - n\,M_{x+n})}{D_x}$$

and $(I\,A)^1_{x\overline{n|}} = \dfrac{R_x - R_{x+n} - n\,M_{x+n}}{D_x}$.

It will be remembered that the formulas for an annuity and a deferred annuity differed only in the numerators, thus: $a_x = \dfrac{N_x}{D_x}$ and ${}_n|a_x = \dfrac{N_{x+n}}{D_x}$. And that a similar difference exists between the formulas for insurance, thus:

$$A_x = \frac{M_x}{D_x} \text{ and } {}_n|A_x = \frac{M_{x+n}}{D_x}.$$

Since S_x is the sum of values of N beginning with N_x and R_x of M beginning with M_x, it is evident that the same relations will exist when S or R is employed, as in the case of varying benefits, as exist when N and M are employed. Thus if the value of a varying annuity or insurance deferred t years be desired, it is only necessary to modify the formulas by using x + t wherever x appears in the *numerators only*.

INTER-RELATIONS BETWEEN SINGLE PREMIUMS.

SUPPOSE an insurance, either for life or n years, payable as an annuity certain for m years, first payment immediate; the value is equivalent to a single premium for an insurance payable in one sum for an amount equal to the discounted or commuted value of the annuity.

Suppose an endowment in n years, payable as an annuity certain for m years, first payment immediate; the value is equivalent to a single premium for an endowment equal to the commuted value of the annuity.

Suppose an endowment insurance for n years, payable as an annuity certain for m years, first payment immediate; the value is again equivalent to a single premium for an endowment insurance of the commuted value.

Combinations may also be made between temporary insurance and endowment, providing either for insurance in a lump sum and endowment in an annuity certain or for insurance in an annuity certain and endowment in a lump sum, by remembering that the value of the element involving an annuity certain is equivalent to a single premium for the commuted value of the annuity.

It has been already pointed out that the value of a life or temporary annuity or insurance deferred n years is equivalent to the single premium of an endowment for an amount equal to the value of the annuity or insurance n years hence.

Even a whole life insurance may be split into a temporary insurance to the age next before the extreme age of the table plus a pure endowment due at the end of the year to all who enter that year of age; or may be considered an endowment insurance, maturing at the extreme age.

ANNUAL PREMIUMS. LIFE AND LIMITED PAYMENT LIFE.

THE value of an annual premium of 1, first payment immediate and payable for life, is plainly the value of an immediate life annuity of 1 or $1 + a_x$. Consequently there will be as many times 1 in an annual premium equivalent to a single premium for whole life insurance as $1 + a_x$ is contained in A_x. Designating the annual premium by P_x, we have, then, the formula:

$$P_x = \frac{A_x}{1 + a_x}$$

To get this formula in terms of A_x only, let us substitute for a_x its equivalent value, found in our conversion formulas, of $\frac{1 - A_x}{d} - 1$; this gives

$$P_x = \frac{A_x}{1 + \frac{1 - A_x}{d} - 1} = \frac{A_x}{\frac{1 - A_x}{d}} = \frac{d\,A_x}{1 - A_x}$$

To get the formula in terms of a_x only, let us substitute for A_x its equivalent from our conversion formulas: $1 - d\,(1 + a_x)$; this gives

$$P_x = \frac{1 - d\,(1 + a_x)}{1 + a_x} = \frac{1}{1 + a_x} - d$$

Conversely, we may derive A_x from P_x, thus

$$P_x = \frac{d\,A_x}{1 - A_x}$$

$$P_x\,(1 - A_x) = d\,A_x$$

$$P_x = A_x\,(P_x + d)$$

$$A_x = \frac{P_x}{P_x + d}$$

And also a_x from P_x, thus:

$$P_x = \frac{1}{1 + a_x} - d$$

$$P_x (1 + a_x) = 1 - d\,(1 + a_x)$$

$$a_x (P_x + d) = 1 - P_x - d$$

$$a_x = \frac{1 - (P_x + d)}{P_x + d} = \frac{1}{P_x + d} - 1.$$

In commutation symbols, the formula for the annual premium for life becomes:

$$P_x = \frac{A_x}{1 + a_x} = \frac{M_x}{D_x} \div \frac{N_{x-1}}{D_x} = \frac{M_x}{N_{x-1}}$$

Or from an alternative formula:

$$P_x = \frac{1}{1 + a_x} - d = \frac{1}{\frac{N_{x-1}}{D_x}} - d = \frac{D_x}{N_{x-1}} - d$$

Or from the other alternative formula:

$$P_x = \frac{d\,A_x}{1 - A_x} = \frac{d\,M_x}{D_x} \div \left(1 - \frac{M_x}{D_x}\right) = \frac{d\,M_x}{D_x - M_x}$$

The value of an annual premium of 1, first payment immediate, limited to t premiums, is the value of an immediate temporary annuity for t years, thus: $1 + {}_{|t-1}a_x$ or $1 + a_{x\,\overline{t-1}|}$. Consequently there will be as many times 1 in the limited annual premium for t years equal to a single premium as the value of the immediate annuity for t years is contained in the single premium, thus:

$${}_tP_x = \frac{A_x}{1 + a_{x\,\overline{t-1}|}}$$

In commutation symbols, this takes the form:

$${}_tP_x = \frac{M_x}{D_x} \div \frac{N_{x-1} - N_{x+t-1}}{D_x} = \frac{M_x}{N_{x-1} - N_{x+t-1}}$$

Note: π is the American symbol for an annual premium; in English notation it often signifies a *special* annual premium. In American notation the formulas are:

$$\pi_x = \frac{\Pi_x}{A_x} = \frac{d\,\Pi_x}{1 - \Pi_x} = \frac{1}{A_x} - d = \frac{M_x}{N_x}$$

$$= \frac{D_x}{N_x} - d = \frac{d\,M_x}{D_x - M_x}$$

$$\pi_{xt} = \frac{\Pi_x}{A_{xt}} = \frac{M_x}{N_x - N_{x+t}}$$

ANNUAL PREMIUMS. TERM, DEFERRED INSURANCES, ANNUITIES AND ENDOWMENTS.

Suppose an insurance for n years with premiums annually in advance. The value of an annuity due for n years has been found to be $1 + a_{x\,\overline{n-1|}}$. Designating the premium by the symbol, $|_{n}P_x$ or $P^{1}_{x\,\overline{n|}}$, we have

$$|_{n}P_x \text{ or } P^{1}_{x\,\overline{n|}} = \frac{|_{n}A_x \text{ or } A^{1}_{x\,\overline{n|}}}{1 + a_{x\,\overline{n-1|}}}$$

In commutation symbols:

$$P^{1}_{x\,\overline{n|}} = \frac{M_x - M_{x+n}}{D_x} \div \frac{N_{x-1} - N_{x+n-1}}{D_x} = \frac{M_x - M_{x+n}}{N_{x-1} - N_{x+n-1}}$$

It may be desired to have this formula in terms of D or N, instead of M. We have seen that:

$$M_x = v\,N_{x-1} - N_x = D_x - d\,N_{x-1}.$$

Substituting, then, these values in our formula:

$$P^{1}_{x\,\overline{n|}} = \frac{v\,N_{x-1} - N_x - (v\,N_{x+n-1} - N_{x+n})}{N_{x-1} - N_{x+n-1}}$$

$$= v - \frac{N_x - N_{x+n}}{N_{x-1} - N_{x+n-1}}$$

Or

$$P^{1}_{x\,\overline{n|}} = \frac{D_x - d\,N_{x-1} - (D_{x+n} - d\,N_{x+n-1})}{N_{x-1} - N_{x+n-1}}$$

$$= \frac{D_x - D_{x+n}}{N_{x-1} - N_{x+n-1}} - d$$

Suppose a pure endowment in n years, annual premiums in advance; we have:

$$P_n\,E_x = \frac{{}_{n}E_x}{1 + a_{x\,\overline{n-1|}}}$$

In commutation symbols:

$$P_n\,E_x = \frac{D_{x+n}}{D_x} \div \frac{N_{x-1} - N_{x+n-1}}{D_x} = \frac{D_{x+n}}{N_{x-1} - N_{x+n-1}}$$

Suppose an endowment insurance for n years, annual premiums in advance; we have:

$$P_n Æ_x \text{ or } P_{x\overline{n|}} = \frac{{}_nÆ_x \text{ or } A_{x\overline{n|}}}{1 + a_{x\,\overline{n-1|}}}$$

In commutation symbols:

$$P_{x\overline{n|}} = \frac{M_x - M_{x+n} + D_{x+n}}{D_x} \div \frac{N_{x-1} - N_{x+n-1}}{D_x}$$

$$= \frac{M_x - M_{x+n} + D_{x+n}}{N_{x-1} - N_{x+n-1}}$$

Of this last formula, $\frac{M_x - M_{x+n}}{N_{x-1} - N_{x+n-1}}$ is the insurance portion and $\frac{D_{x+n}}{N_{x-1} - N_{x+n-1}}$ the pure endowment portion. It may be desired to have this formula in terms of D and N only; if so, substituting the value $\frac{D_x - D_{x+n}}{N_{x-1} - N_{x+n-1}} - d$ for the equivalent $\frac{M_x - M_{x+n}}{N_{x-1} - N_{x+n-1}}$, we have:

$$P_{x\,\overline{n|}} = \frac{D_x - D_{x+n}}{N_{x-1} - N_{x+n-1}} - d + \frac{D_{x+n}}{N_{x-1} - N_{x+n-1}}$$

$$= \frac{D_x}{N_{x-1} - N_{x+n-1}} - d$$

Similarly by substituting, instead, $v - \frac{N_x - N_{x+n}}{N_{x-1} - N_{x+n-1}}$ we obtain:

$$P_{x\overline{n|}} = v - \frac{N_x - N_{x+n}}{N_{x-1} - N_{x+n-1}} + \frac{D_{x+n}}{N_{x-1} - N_{x+n-1}}$$

$$= v - \frac{N_x - (N_{x+n} + D_{x+n})}{N_{x-1} - N_{x+n-1}}$$

Suppose an insurance for n years, annual premiums in advance for t years only; we have:

$$P^1_{x\overline{n|}} = \frac{A^1_{x\overline{n|}}}{1 + a_{x\,\overline{t-1|}}}.$$

In commutation symbols:

$$ {}_tP^1_{x\overline{n}|} = \frac{M_x - M_{x+n}}{D_x} \div \frac{N_{x-1} - N_{x+\overline{t-1}}}{D_x} $$
$$ = \frac{M_x - M_{x+n}}{N_{x-1} - N_{x+t-1}} $$

Suppose an endowment insurance in n years, annual premiums in advance for t years; we have:

$$ {}_tP_{x\overline{n}|} = \frac{A_{x\overline{n}|}}{1 + a_{x\overline{t-1}|}} $$

In commutation symbols:

$$ {}_tP_{x\overline{n}|} = \frac{M_x - M_{x+n} + D_{x+n}}{D_x} \div \frac{N_{x-1} - N_{x+\overline{t-1}}}{D_x} $$
$$ = \frac{M_x - M_{x+n} + D_{x+n}}{N_{x-1} - N_{x+t-1}} $$

Suppose an insurance deferred n years, annual premiums in advance for n years only; we have:

$$ {}_nP_n|A_x = \frac{{}_n|A_x}{1 + a_{x\overline{n-1}|}} $$

In commutation symbols:

$$ {}_nP_n|A_x = \frac{M_{x+n}}{D_x} \div \frac{N_{x-1} - N_{x+\overline{n-1}}}{D_x} $$
$$ = \frac{M_{x+n}}{N_{x-1} - N_{x+n-1}} $$

If paid for by t annual premiums, these formulas become:

$$ {}_tP_n|A_x = \frac{{}_n|A_x}{1 + a_{x\overline{t-1}|}} = \frac{M_{x+n}}{N_{x-1} - N_{x+t-1}} $$

Suppose an insurance for m years deferred n years, annual premiums for n + m years:

$$ {}_nP_n|{}_mA_x = \frac{{}_n|{}_mA_x}{1 + a_{x\overline{n-1}|}} = \frac{M_{x+n} - M_{x+n+m}}{N_{x-1} - N_{x+n-1}} $$

Or if paid by t annual premiums:

$$ {}_tP_n|{}_mA_x = \frac{{}_n|{}_mA_x}{1 + a_{x\overline{t-1}|}} = \frac{M_{x+n} - M_{x+n+m}}{N_{x-1} - N_{x+t-1}} $$

Suppose an annuity deferred n years, annual premiu in advance; we have:

$$P_{n}|a_x = \frac{{}_{n|}a_x}{1 + a_{x\,\overline{n-1|}}} = \frac{N_{x+n}}{N_{x-1} - N_{x+n-1}}$$

If paid for by t annual premiums:

$${}_{t}P_{n}|a_x = \frac{{}_{n|}a_x}{1 + a_{x\,\overline{t-1|}}} = \frac{N_{x+n}}{N_{x-1} - N_{x+t-1}}$$

Suppose an annuity for m years, deferred n years, ann premiums in advance for n years; we have:

$$P_{n}|{}_{m}a_x = \frac{{}_{n|m}a_x}{1 + a_{x\,\overline{n-1|}}} = \frac{N_{x+n} - N_{x+n+m}}{N_{x-1} - N_{x+n-1}}$$

If paid for by t annual premiums:

$${}_{t}P_{n}|{}_{m}a_x = \frac{{}_{n|m}a_x}{1 + a_{x\,\overline{t-1|}}} = \frac{N_{x+n} - N_{x+n+m}}{N_{x-1} - N_{x+t-1}}$$

Note: In American notation, the most important f mulas are:

$$t\,\pi_{x\,n} = \frac{\Pi_{x\,n}}{A_{x\,n}} = \frac{M_x - M_{x+n}}{N_x - N_{x+n}} = v - \frac{N_{x+1} - N_{x+n+}}{N_x - N_{x+n}}$$

$$= \frac{D_x - D_{x+n}}{N_x - N_{x+n}} - d$$

$${}_{s}\pi_{x\,n} = \frac{{}_{s}\Pi_{x\,n}}{A_{x\,n}} = \frac{D_{x+n}}{N_x - N_{x+n}}$$

$${}_{e}\pi_{x\,n} = \frac{{}_{e}\Pi_{x\,n}}{A_{x\,n}} = \frac{M_x - M_{x+n} + D_{x+n}}{N_x - N_{x+n}}$$

$${}_{e}\pi_x\,t_n = \frac{{}_{e}\Pi_{x\,n}}{A_{x\,t}} = \frac{M_x - M_{x+n} + D_{x+n}}{N_x - N_{x+t}}$$

ANNUITIES PAYABLE FRACTIONALLY. CONTINUOUS ANNUITIES.

Let $a_x^{(m)}$ be the symbol for an annuity, payable 1/m at the end of each mth part of a year.

Let there be m annuities, with first payments of 1, at the end of 1/mth year, 2/mth year, etc. The sum of their values is, approximately:

$$\left(a_x + \frac{m-1}{m}\right) + \left(a_x + \frac{m-2}{m}\right) + \ldots\ldots \left(a_x + \frac{m-m}{m}\right)$$

$$= m\,a_x + \frac{m(m-1)}{2\,m}.$$

Dividing by m,

$$a_x^{(m)} = a_x + \frac{m-1}{2\,m} \text{ and } \mathrm{a}_x^{(m)} = a_x^{(m)} + 1/m = a_x + \frac{m+1}{2\,m}$$

This formula for $a_x^{(m)}$ is approximate only, but is the one usually employed, whether for annuities upon one life, joint lives, or joint and survivor.

For annuities payable each half year, the usual approximation is:

$$a_x^{(2)} = a_x + 1/4 \text{ and } \mathrm{a}_x^{(2)} = a_x^{(2)} + 1/2 = a_x + 3/4.$$

Payable four times a year:

$$a_x^{(4)} = a_x + 3/8 \text{ and } \mathrm{a}_x^{(4)} = a_x^{(4)} + 1/4 = a_x + 5/8.$$

If m be made to approach infinity, $\frac{m-1}{2\,m}$ approaches $\frac{1}{2}$ and, assigning $\overline{a}_x$ to mean an annuity of 1 per annum accruing in momently instalments, known as a continuous annuity, we have:

$$\overline{a}_x = a_x + 1/2.$$

The value of a deferred annuity, payable m times a year, may be found by:

$${}_{t|}a_x^{(m)} = {}_tE_x\left(a_{x+t}^{(m)}\right)$$

$${}_{t|}\mathrm{a}_x^{(m)} = {}_tE_x\left(\mathrm{a}_{x+t}^{(m)}\right)$$

And the value of a temporary annuity, payable m times a year, by the relation,

$$|{}_t a^{(m)}_x = a^{(m)}_x - {}_t|a^{(m)}_x.$$

$$|{}_t \mathrm{a}^{(m)}_x = \mathrm{a}^{(m)}_x - {}_t|\mathrm{a}^{(m)}_x.$$

So likewise, as to continuous annuities:

$${}_t|\bar{a}_x = {}_tE_x\,(\bar{a}_{x+t}).$$

$$|{}_t\bar{a}_x = \bar{a}_x - {}_t|\bar{a}_x.$$

An annuity due, with first payment deferred 1/tth part of a year, may be considered roughly equivalent to an annuity with first payment at the end of the year, plus an immediate payment of $1 - \frac{1}{t} = \frac{t-1}{t}$; that is:

$${}_{\frac{1}{t}}|\mathrm{a}_x = a_x + \frac{t-1}{t} = \mathrm{a}_x - 1/t$$

$$\tfrac{1}{2}|\mathrm{a}_x = a_x + \tfrac{1}{2} = \mathrm{a}_x - \tfrac{1}{2}.$$

When a temporary annuity has its first payment due in 1/tth part of a year, instead of at the end of the year, the effect is nearly the same as if there were $\frac{t-1}{t}$ paid down, an annuity of 1 at the end of the year for n—1 years and a payment of 1/t at the end of n years; that is:

$${}_{\frac{1}{t}}|\mathrm{a}_{x\,\overline{n|}} = a_{x\,\overline{n-1|}} + \frac{t-1}{t} + \frac{1}{t}\,{}_nE_x.$$

Or its value may be found approximately from the approximate value of a deferred annuity:

$${}_{n+\frac{1}{t}}|a_x = {}_nE_x\left(a_x + \frac{t-1}{t}\right).$$

Whence $1/t|\mathrm{a}_{x\,\overline{n|}} = 1/t|\mathrm{a}_x - {}_{n+1/t}|\mathrm{a}_x$.

The value of a continuous annuity deferred 1/tth part of a year, approximately:

$${}_{\frac{1}{t}}|\bar{a}_x = \bar{a}_x - 1/t$$

$${}_{n+\frac{1}{t}}|\bar{a}_x = {}_nE_x\,(\bar{a}_x - 1/t)$$

$${}_{\frac{1}{t}}|\bar{a}_{x\,\overline{n|}} = {}_{\frac{1}{t}}|\bar{a}_x - {}_{n+\frac{1}{t}}|\bar{a}_x.$$

PREMIUMS PAYABLE FRACTIONALLY.

In the foregoing chapters on computation of annual premiums, it has been assumed that all premiums are paid annually in advance.

In the United States all premiums are usually treated as so payable; and when semi-annual or quarterly payment is preferred, the payments are treated as instalments of the annual premium and are deducted upon payment of a death-claim or other settlement.

The usual custom in the United States is to add 4% to the annual premium and divide by two, to get the semi-annual premium; and to add 6% and divide by four, to get the quarterly premium.

Part of the additional charge is to cover interest which is lost and part to cover the extra expense incurred. Since the unpaid portion of the premium is deducted at death, no charge need be made because of an increase of the average actual insurance or amount at risk.

The net annual premium, in order to cover the loss of interest when premiums are paid in semi-annual instalments, should be increased by $\frac{i}{4}$ upon itself and be divided by two; and when paid quarterly, be increased by $\frac{3}{8}i$ upon itself and be divided by four.

In Great Britain, the practice is quite usual of quarterly and semi-annual premiums being complete in themselves, no deduction being made for unpaid portions of the year's premium in paying a death-claim. The addition to the gross annual premium is usually about 3% for semi-annual premiums, and 5% or 6% for quarterly.

The true net semi-annual and quarterly premiums, however, may be computed by means of annuities due, payable twice and four times a year, respectively. That is:

$$P_x^{(2)} = \frac{A_x}{\frac{3}{4} + a_x}.$$

$$P_x^{(4)} = \frac{A_x}{\frac{5}{8} + a_x}.$$

In the United States, monthly premium policies are sometimes written one way and sometimes the other. The gross annual premium is usually increased from 20 per cent to 32 per cent—a common rule being to take 10 per cent or 11 per cent or even 12 per cent of the gross annual premium for the monthly premium,

When the monthly premium is an instalment, the unpaid portion of the annual premium being deducted at death, the net annual premium should be increased by $\frac{33}{72}$ i upon itself, though $\frac{i}{2}$ will answer if strict accuracy be not required.

True net monthly premiums, no deduction being made at death, are found by the formula:

$$P_x^{(12)} = \frac{A_x}{\ddot{a}_x^{(12)}} = \frac{A_x}{a_x + \frac{13}{24}} = \frac{A_x}{\ddot{a}_x - \frac{11}{24}}.$$

In practice, however, true net monthly premiums frequently and true net weekly premiums always are taken as $\frac{1}{12}$ and $\frac{1}{52}$, respectively, of the net annual continuous premium; that is:

$$P_x^{(12)} = \frac{\bar{P}_x}{12}$$

$$P_x^{(52)} = \frac{\bar{P}_x}{52}$$

Weekly premiums, as employed in industrial insurance, are always true premiums and not instalments of an annual premium. There is no deduction at death on account of the unpaid portion of the annual premium.

VARYING PREMIUMS.

Let us have an insurance of 1 payable at death with a net annual premium of π for the first t years, increasing or decreasing by k each period of t years thereafter for m such periods and then level at $\pi \pm m k$.

$$A_x = \pi a_x \pm k\,(t|a_x + {}_{2t}|a_x + \ldots \; {}_{mt}|a_x)$$

$$M_x = \pi N_{x-1} \pm k\,(N_{x+t-1} + N_{x+2t-1} + \ldots \; N_{x+mt-1})$$

$$\pi = \frac{M_x \mp k(N_{x+t-1} + N_{x+2t-1} + \ldots \; N_{x+mt-1})}{N_{x-1}}$$

Note, that when the premium is increasing π should not in practice be less than $P^1_{x\,\overline{1}|}$ nor should $\pi + m k$ be greater than P_{x+mt}. In the former case the insurance might be discontinued at the end of the first year to the disadvantage of the company, which would not have received sufficient premium from which to meet the first year's risk; and, in the latter, the assured if still healthy after m t years could take his insurance for the remainder of life at the rate of P_{x+mt}. When the premium is decreasing, if $\pi < m k$, it means that after π is reduced to zero, an annuity is payable under the policy.

Let us have an insurance with a net annual premium of π for the first t years, π_1 for the second t years, etc., to π_m for life after m t years, all the premiums, except π_m, being known; to determine π_m, we have:

$$A_x = \pi a_x + (\pi_1 - \pi)\,{}_{t}|a_x + (\pi_2 - \pi_1)\,{}_{2t}|a_x + \ldots \; (\pi_m - \pi_{m-1})\,{}_{mt}|a_x$$

$$M_x = \pi N_{x-1} + (\pi_1 - \pi)N_{x+t-1} + (\pi_2 - \pi_1)N_{x+2t-1} + \ldots$$
$$(\pi_m - \pi_{m-1})\,N_{x+mt-1}$$

$$\pi_m = \frac{M_x - [\pi N_{x-1} + (\pi_1 - \pi)N_{x+t-1} + \ldots \; (\pi_{m-1} - \pi_{m-2}) N_{x+\overline{m-1}t-1} - \pi_{m+1} N_{x+mt-1}]}{N_{x+mt-1}}$$

The most common form, perhaps, of increasing or decreasing premium is one which increases or decreases by a percentage of itself. Let us have an insurance with an annual premium of π, the first t years, increasing or decreasing by k π every t years, for m t years and thereafter level at π (1 ± m k).

$$A_x = \pi a_x \pm k \pi ({}_{t|}a_x + {}_{2t|}a_x \dots \; {}_{mt|}a_x)$$

$$M_x = \pi N_{x-1} \pm k \pi (N_{x+t-1} + \dots \; N_{x+mt-1})$$

$$\pi = \frac{M_x}{N_{x-1} \pm k (N_{x+t-1} + \dots \; N_{x+mt-1})}$$

A plan of insurance, popular in Great Britain, is that known as the "half premium" plan. The premium during the first t years is but one-half of that charged during the rest of life. Let $\pi =$ net premium during first t years, $2\pi =$ net premium thereafter.

$$A_x = \pi a_x + \pi {}_{t|}a_x .$$

$$M_x = \pi N_{x-1} + \pi N_{x+t-1}.$$

$$\pi = \frac{M_x}{N_{x-1} + N_{x+t-1}}.$$

INSURANCES PAYABLE OTHERWISE THAN AT END OF YEAR.

Theoretically all insurances that have been considered so far, are payable at the end of the year in which death may occur, that is, on the average, six months after death.

In practice, so far as the United States is concerned, death-losses are paid immediately upon receipt of proofs, or nearly so; premiums are, notwithstanding, computed on the other assumption and reserves are made accordingly.

It is desirable to investigate, then, the nature and the effect of the assumption.

Suppose, for instance, that claims were to be paid at the end of periods of six months. On the assumption that deaths take place at equal intervals over the six months it would be three months, on the average, from the date of death until payment of the claim. It is evident, that on the average, three months' interest would be lost on every claim, as compared with those payable at the end of the year; that is, on an average, *six* months after death. The single premium, therefore, should be:

$$A^{(2)}_x = A \ (1+i)^{\frac{1}{4}} = A_x \ (1+\tfrac{i}{4}) \text{ approx.}$$

Similarly, if claims were supposed to be paid at the end of periods of $1/m^{th}$ of a year, interest would be lost for a period of $\left(\frac{1}{2} - \frac{1}{2\,m}\right)$ year $= \frac{m-1}{2\,m}$ year; so that:

$$A^{(m)}_x = A_x \ (1+i)^{\frac{m-1}{2m}} = A_x \left(1 + \frac{m-1}{2\,m}\, i\right) \text{approx.}$$

And if claims are payable at the moment of death, we have:

$$\bar{A}_x = A_x \ (1+i)^{\frac{1}{2}} = A_x \left(1 + \frac{i}{2}\right) \text{ approx.}$$

The same ratios obtain between insurances of the ordinary form and of these special forms, in the cases of temporary and deferred insurances.

The case of an endowment insurance differs in that the endowment is payable at the end of n years, in that regard the contract remaining unchanged; so that, in order to compute the value when death-claims are payable otherwise than at the end of the year, we must first separate the endowment insurance into a temporary insurance and a pure endowment. The value, then, becomes:

$$A^{(m)}_{\overset{1}{x}\,\overline{n}|} + A_{x\,\overset{1}{\overline{n}|}} = A_{\overset{1}{x}\,\overline{n}|}\left(1 + \frac{m-1}{2m}\,i\right) + A_{x\,\overset{1}{\overline{n}|}}.$$

$$\overline{A}_{\overset{1}{x}\,\overline{n}|} + A_{x\,\overset{1}{\overline{n}|}} = A_{\overset{1}{x}\,\overline{n}|}\left(1 + \frac{i}{2}\right) + A_{x\,\overset{1}{\overline{n}|}}.$$

These forms of insurance correspond to the annuities dealt with in the last chapter and may be directly derived therefrom, thus:

$$A^{(2)}_{x} = 1 - f_{(2)}\,a^{(2)}_{x}$$

$$A^{(m)}_{x} = 1 - f_{(m)}\,a^{(m)}_{x}$$

$$\overline{A}_{x} = 1 - \delta\bar{a}_{x}$$

Annual premiums, however, are computed by the use of ordinary annuities due; and the annuities in these single premium formulas are used only when premiums payable twice a year or m times a year or continuously throughout the year, are desired.

COMPLETE ANNUITIES.

All the annuities so far considered have been of the form known as "curtate"; that is, no part is payable for the year (or fraction of a year, as the case may be) in which death occurs, the last payment being at the end of the last complete year (or complete fraction of a year) that the life survives. Annuities of this form are much more common than any other; it is the only form used in the computation of annual premiums.

But it is also possible to have "complete annuities" under which a part of the regular payment, proportionate to the time elapsed since the last annuity payment, is due at the date of death. Such annuities are called "complete" or "apportionable." By the laws of Great Britain, all annuities, unless otherwise specified, are complete.

Continuous annuities by their nature are already complete.

If an annuity for the life of x be complete, it will involve a payment of ½, on the average, at the end of the year in which x may die. Its value, therefore, is:

$$\overset{\circ}{a}_x = a_x + \frac{A_x}{2}$$

Or if, as is usual, the payment be due immediately upon the death of x, more accurately:

$$\overset{\circ}{a}_x = a_x + \frac{\bar{A}_x}{2} = a_x + \frac{A_x\,(1+\frac{i}{2})}{2}$$

Similarly:

$$\overset{\circ}{a}{}^{(m)}_x = a^{(m)}_x + \frac{A_x\,(1+\frac{i}{2})}{2\,m}.$$

For deferred, use the formulas:

$$ {}_{t|}\overset{\circ}{a}_x = {}_tE_x\,\overset{\circ}{a}_{x+t}$$

$$ {}_{t|}\overset{\circ}{a}{}^{(m)}_x = {}_tE_x\,\overset{\circ}{a}{}^{(m)}_{x+t}$$

And for temporary, the formulas:

$$|_{t}\mathring{a}_x = \mathring{a}_x - {}_{t}|\mathring{a}_x .$$

$$|_{t}\mathring{a}^{(m)}_{x} = \mathring{a}^{(m)}_{x} - {}_{t}|\mathring{a}^{(m)}_{x}$$

For temporary annuities the results may also be derived thus:

$$\begin{aligned} |_{t}\mathring{a}_x &= |_{t}a_x + \tfrac{1}{2}\, A^{1}_{x\,\overline{t|}} \\ &= |_{t}a_x + \tfrac{1}{2}\, A^{1}_{x\,\overline{t|}}\left(1 + \tfrac{i}{3}\right) \end{aligned}$$

$$|_{t}\mathring{a}^{(m)}_{x} = a^{(m)}_{x} + \tfrac{1}{2m}\, A^{1}_{x\,\overline{t|}}\left(1 + \tfrac{i}{3}\right)$$

LIFE ANNUITY VERSUS ANNUITY FOR EXPECTANCY.

A common error of the tyro in actuarial science is to suppose that a life annuity is an annuity for a term certain, equal to the expectation of life. The error is an old one. The Romans apparently computed the value of a life annuity in some such fashion; and, so late as the 18th century, a thick volume was published to prove the incorrectness of the method, described in the foregoing, because it brought out values differing from the values of annuities certain for the expectation of life and also differing from such values by different ratios, according to the rate of interest employed.

It is possible to prove algebraically that the value of a life annuity is always smaller than the value of an annuity for the expectancy.

$$\text{Let } e_x = n + t$$

in which n is integral and t fractional.

We have:

$$a_x = v\,p_x + v^2\,{}_2p_x + v^3\,{}_3p_x + \ldots\ldots$$

$$a_{\overline{n+t|}} = v + v^2 + v^3 + \ldots\ldots\ v^n + t\,v^{n+1}$$

in which latter we assume that the partial payment for t part of a year, is paid at the end of the year. If it be assumed that it is paid at once upon the expiration of $n + t$ years, the value of $a_{\overline{n+t|}}$ would be slightly greater.

Whether

(1.) $v\,p_x + v^2\,{}_2p_x + \ldots\ldots < \text{ or } > v + v^2 + \ldots + v^n + t\,v^{n+1}$

may be determined by deducting $v\,p_x + v^2\,{}_2p_x + \ldots\ldots + v^{n+1}\,{}_{n+1}p_x$ from both sides and observing which of the remainders is the larger. Thus, whether:

$(2.) v^{n+2}\,{}_{n+2}p_x + v^{n+3}\,{}_{n+3}p_x \;.\;.\;.\;.\;.\; < \text{or} > v\,(1 - p_x)$
$v^2(1 - {}_2p_x) + \;.\;.\;.\;.\;.\; v^{n+1}\,(t - {}_{n+1}p_x)$

But

$v^{n+2}\,{}_{n+2}p_x + v^{n+3}\,{}_{n+3}p_x + \;.\;.\;.\;.\;.\; < v^{n+2}\,({}_{n+2}p_x + {}_{n+3}p_x + \;.\;.\;.\;.\;)$

While

$v\,(1 - p_x) + v^2\,(1 - {}_2p_x) + \;.\;.\;.\;.\;.\; v^{n+1}(t - {}_{n+1}p_x)$
$> v^{n+1}[\,(1 - p_x) + (1 - {}_2p_x) + \;.\;.\;.\;.\;.\; (t - {}_{n+1}p_x)\,]$
$> v^{n+1}\,[n+t - (p_x + {}_2p_x + \;.\;.\;.\;.\;.\; {}_{n+1}p_x)\,]$

But

$n+t = e_x = p_x + {}_2p_x + {}_3p_x + \;.\;.\;.\;.\;.$

So that

$v\,(1 - p_x) + v^2\,(1 - {}_2p_x) + \;.\;.\;.\;.\; v^{n+1}\,(t - {}_{n-1}p_x)$
$> v^{n+1}\,[\,(p_x + {}_2p_x + {}_3p_x + \;.\;.\;.\;.\;.\;) - (p_x + {}_2p_x \;.\;.\;.\; {}_{n+1}p_x)]$
$> v^{n+1}\,({}_{n+2}p_x + {}_{n+3}\,p_x + {}_{n+4}p_x + \;.\;.\;.\;.\;.\;)$

But

$(3.)\; v^{n+1}\,({}_{n+2}p_x + {}_{n+3}p_x + \;.\;.\;.\;.\;) > v^{n+2}\,({}_{n+2}p_x + {}_{n+3}p_x + \;.\;.\;.)$

Therefore, since the remainder on the left side of (2) is less than the right side of (3) and the remainder on the left side of (2) is more than the left side of (3) and the left side of (3) is more than the right side of (3), the condition holds that:

$v\,p_x + v^2\,{}_2p_x + v^3\,{}_3p + \;.\;.\;.\; < v + v^2 + v^3 + \;.\;.\;.\; t\,v^{n+1}$

That is:

$a_x < a_{\overline{e_x}|}$

ANNUAL OFFICE OR GROSS PREMIUMS. THE LOADING.

In considering single premiums we learned that to make the office or gross premium it is customary to add to the net premium either (1st), a percentage of itself, $(A + k A)$ or, (2d), a fixed sum, $(A + c)$ or, (3d) both percentage and constant, thus: $A + k A + c$ or $A(1 + k) + c$. And we found that the last formula will answer for all three cases; for when $c = 0$, it becomes identical with the first, and when $k = 0$ with the second.

The custom in adding "loading" to annual premiums is similar; such gross premiums taking the form, respectively: $P(1 + k)$, $P + c$ or $P(1 + k) + c$, the last again comprehending both the others.

An analysis of the subject of "loading" so as to arrive at more exact conceptions of how it should be done may not be out of place in this connection; and to understand the subject at all, it is necessary to consider the purposes for which this extra contribution is levied. The main purposes are (1st) to provide a contingent fund to cover unforeseen losses of any sort, and (2d) to meet the expenses.

As to the first of these, the better system is now generally acknowledged to be to construct the net premium on such mortality and interest hypotheses as will guarantee its sufficiency and in addition to accumulate a surplus to cover contingent losses.

The most important thing, then, to be considered is the expense, and usually attention is directed almost exclusively to that. Expenses may be divided into several distinct categories, such as: cost of new business, cost of collection and renewal, cost of general management and cost of care of investments.

These costs commonly come in different forms and at different times. The cost of new business comes but once, viz., at the issue of policies. It comprises commissions for new insurances, medical fees, agency salaries and expenses, canvassing literature and supplies, etc. The largest item, generally, is for commissions, which are paid mostly as percentages on first premiums. It is, therefore, not possible to escape the conclusions: first, that the expense comes in a lump sum at the outset, and, second, mainly as a percentage charge on premiums.

The cost of collection and renewal also comes in the form of a percentage charge on premiums after the first.

The general management of a life insurance company is in order to furnish insurance, and it appears clear that it should be a fixed charge per $1000 insurance carried, or at most should be proportionate, not to the premium, but to the costs of insurance which are the *values* of the benefits currently furnished.

The cost of caring for investments is properly chargeable against the income from investments and need not appear in the loading. It should appear in determining the interest factor in constructing net premiums and in computing dividends, as only the probable *net* revenue from investments should be considered.

These leave us with three items of widely different nature to incorporate into the loading, viz., cost of new business, coming at the outset and being a lump sum and usually a percentage of the first premium; cost of collection and renewal, a percentage of subsequent premiums; and cost of insurance management, a fixed sum per $1000 insured.

Concerning the first of these, it must be premised that the first premiums must be large enough to meet this cost and the mortality cost of the year or else moneys contributed by other insurances will be trenched upon. It has been attempted to meet the difficulty in two ways, viz., first, by charging a larger premium the first year to cover the additional expense; second, by charging the same or a smaller gross premium the first year, but treating it as covering only one year term insurance, the other insurance with its net premium beginning the next year at one year's advance of age. The first plan is difficult to sell and the second has not demonstrated its success.

Another plan has been adopted and employed by French companies whose actuaries have devoted a deal of thought to the subject. This is to determine the amount of money required to cover this cost; to deduct this amount from the net premium for the first year and to add its equivalent value in an annuity to each net premium, including the first year. Designating this cost by F and the corresponding annuity by f, we have for a life policy:

Net premium after 1st year ${}^{2}P_x = P_x + f$

Net premium 1st year ${}^{1}P_x = P_x + f - F$

This plan appears to answer all the requirements.

The cost of collection and renewal offers no difficulties, being a function, which we will designate as k, of the premium after the first year; but as it will really need to be a function of the *gross* premium, we defer it for a moment.

The cost of management may be added to the net premium as a fixed amount (c), each year including the first.

The values of P_x, F, f and c are, then, determined in advance, and we have the following requirements established:

${}^{1}P'_x$ (gross premium 1st year) $= P'_x$ (gross premium thereafter).

${}^2P_x$ (net premium after 1st year) $= P_x + f$

${}^1P_x$ (net premium 1st year) $= {}^2P_x - F = P_x + f - F$

$F = f\,(1 + a_x)$.

From these, remembering that the premium when loaded with other costs must be loaded by k per cent of itself for cost of collection and renewal, we get:

$P'_x = ({}^2P_x + c)\,(1 + k) = (P_x + f + c)\,(1 + k)$

${}^1P'_x = ({}^1P_x + F + c)\,(1 + k) = ({}^2P_x + c)\,(1 + k) = (P_x + f + c)\,(1 + k)$

When net premiums are made by an ultimate table and reserves are computed by this net premium but by a select table otherwise, the effect is to allow the present value of all the salvage from mortality, by the select table over the ultimate table, as an offset to initial expenses.

ENDOWMENTS AND INSURANCES WITH RETURN OF NET ANNUAL PREMIUMS, CONTINUOUS THROUGHOUT TERM.

SUPPOSE an insurance for life with return of all net annual premiums. Designate the net annual premium by the symbol π, used in the English notation to signify any *special or unusual annual premium.* The value of this insurance is plainly separable into two portions, an insurance of 1 and an insurance of π, increasing yearly by π or, in symbols, $A_x + \pi\,(IA)_x$ and the annual premium may be found by dividing this value by $1 + a_x$, the value of an annuity due, thus:

$$\pi = \frac{A_x + \pi (I\,A)_x}{1 + a_x}$$

$$\pi (1 + a_x) = A_x + \pi (I\,A)_x$$

$$\pi [1 + a_x - (I\,A)_x\,] = A_x$$

$$\pi = \frac{A_x}{1 + a_x - (I\,A)_x}$$

In commutation symbols this takes the form:

$$\pi = \frac{A_x + \pi (I\,A)_x}{1 + a_x}$$

$$= \frac{M_x + \pi R_x}{D_x} \div \frac{N_{x-1}}{D_x} = \frac{M_x + \pi R_x}{N_{x-1}}$$

$$\pi N_{x-1} = M_x + \pi R_x$$

$$\pi (N_{x-1} - R_x) = M_x$$

$$\pi = \frac{M_x}{N_{x-1} - R_x}$$

Suppose a pure endowment due in n years with return of all net annual premiums if death occurs within n years. The value of this is ${}_nE_x$ plus $\pi (I\,A)^1_{x\overline{n}|}$ and the premium may be found by dividing by $1 + a_{x\overline{n-1}|}$, thus:

$$\pi = \frac{{}_nE_x + \pi (IA)^1_{x\,\overline{n|}}}{1 + a_{x\,\overline{n-1|}}}$$

$$\pi (1 + a_{x\,\overline{n-1|}}) = {}_nE_x + \pi (IA)^1_{x\,\overline{n|}}$$

$$\pi [1 + a_{x\,\overline{n-1|}} - (IA)^1_{x\,\overline{n|}}] = {}_nE_x$$

$$\pi = \frac{{}_nE_x}{1 + a_{x\,\overline{n-1|}} - (IA)^1_{x\,\overline{n|}}}$$

In commutation symbols this takes the form:

$$\pi = \frac{{}_nE_x + \pi (IA)^1_{x\,\overline{n|}}}{1 + a_{x\,\overline{n-1|}}}$$

$$= \frac{D_{x+n} + \pi (R_x - R_{x+n} - n\,M_{x+n})}{D_x} \div \frac{N_{x-1} - N_{x+n-1}}{D_x}$$

$$= \frac{D_{x+n} + \pi (R_x - R_{x+n} - n\,M_{x+n})}{N_{x-1} - N_{x+n-1}}$$

$$\pi (N_{x-1} - N_{x+n-1}) = D_{x+n} + \pi (R_x - R_{x+n} - n\,M_{x+n})$$

$$\pi [N_{x-1} - N_{x+n-1} - (R_x - R_{x+n} - n\,M_{x+n})] = D_{x+n}$$

$$\pi = \frac{D_{x+n}}{N_{x-1} - N_{x+n-1} - (R_x - R_{x+n} - n\,M_{x+n})}$$

Suppose an insurance for n years with return of net annual premiums. The value is composed of $A^1_{x\,\overline{n|}} + \pi (IA)^1_{x\,\overline{n|}}$ and the premium is found by dividing by $1 + a_{x\,\overline{n-1|}}$ thus:

$$\pi = \frac{A^1_{x\,\overline{n|}} + \pi (IA)^1_{x\,\overline{n|}}}{1 + a_{x\,\overline{n-1|}}}$$

$$\pi (1 + a_{x\,\overline{n-1|}}) = A^1_{x\,\overline{n|}} + \pi (IA)^1_{x\,\overline{n|}}$$

$$\pi [1 + a_{x\,\overline{n-1|}} - (IA)^1_{x\,\overline{n|}}] = A^1_{x\,\overline{n|}}$$

$$\pi = \frac{A^1_{x\,\overline{n|}}}{1 + a_{x\,\overline{n-1|}} - (IA)^1_{x\,\overline{n|}}}$$

Or in commutation symbols:

$$\pi = \frac{A^1_{x\,\overline{n|}} + \pi (IA)^1_{x\,\overline{n|}}}{1 + a_{x\,\overline{n-1|}}}$$

$$= \frac{M_x - M_{x+n} + \pi (R_x - R_{x+n} - n\,M_{x+n})}{D_x} \div \frac{N_{x-1} - N_{x+n-1}}{D_x}$$

$$= \frac{M_x - M_{x+n} + \pi (R_x - R_{x+n} - n\,M_{x+n})}{N_{x-1} - N_{x+n-1}}$$

$$\pi (N_{x-1} - N_{x+n-1}) = M_x - M_{x+n} + \pi (R_x - R_{x+n} - n\,M_{x+n})$$

$$\pi[N_{x-1} - N_{x+n-1} - (R_x - R_{x+n} - n\,M_{x+n})] = M_x - M_{x+n}$$

$$\pi = \frac{M_x - M_{x+n}}{N_{x-1} - N_{x+n-1} - (R_x - R_{x+n} - n\,M_{x+n})}$$

Suppose an endowment insurance, with return of all net annual premiums either at death or maturity. This consists of an endowment of $1 + n\,\pi$ and of an insurance for n years of $1 + \pi$, increasing annually by π, the value being $(1 + n\pi)\,{}_nE_x + A^1_{x\overline{n}|} + \pi(I\,A)^1_{x\overline{n}|}$ and the premium is found by dividing by $1 + a_{x\,\overline{n-1}|}$, thus:

$$\pi = \frac{(1 + n\,\pi)\,{}_nE_x + A^1_{x\overline{n}|} + \pi(I\,A)^1_{x\overline{n}|}}{1 + a_{x\,\overline{n-1}|}}$$

$$= \frac{(1 + n\pi)\,D_{x+n} + M_x - M_{x+n} + \pi\,(R_x - R_{x+n} - n\,M_{x+n})}{D_x}$$

$$\div \frac{N_{x-1} - N_{x+n-1}}{D_x}$$

$$= \frac{(1 + n\,\pi)\,D_{x+n} + M_x - M_{x+n} + \pi(R_x - R_{x+n} - n\,M_{x+n})}{N_{x-1} - N_{x+n-1}}$$

$$\pi(N_{x-1} - N_{x+n-1}) = (1 + n\,\pi)\,D_{x+n} + M_x - M_{x+n} + \pi(R_x - R_{x+n} - n\,M_{x+n})$$

$$\pi[N_{x-1} - N_{x+n-1} - n\,D_{x+n} - (R_x - R_{x+n} - n\,M_{x+n})] = D_{x+n} + M_x - M_{x+n}$$

$$\pi = \frac{D_{x+n} + M_x - M_{x+n}}{N_{x-1} - N_{x+n-1} - n\,D_{x+n} - (R_x - R_{x+n} - n\,M_{x+n})}$$

Suppose an endowment insurance for n years with all net annual premiums returned only in case of death. The value is then composed of $A_{x\overline{n}|}$ or ${}_n\text{Æ}_x + \pi\,(I\,A)^1_{x\overline{n}|}$ and the annual premium is found by dividing by $1 + a_{x\,\overline{n-1}|}$, thus:

$$\pi = \frac{A_{x\overline{n}|} + \pi(I\,A)^1_{x\overline{n}|}}{1 + a_{x\,\overline{n-1}|}}$$

$$= \frac{M_x - M_{x+n} + D_{x+n} + \pi(R_x - R_{x+n} - n\,M_{x+n})}{D_x} \div$$

$$\frac{N_{x-1} - N_{x+n-1}}{D_x}$$

$$= \frac{M_x - M_{x+n} + D_{x+n} + \pi(R_x - R_{x+n} - n\,M_{x+n})}{N_{x-1} - N_{x+n-1}}$$

$$\pi(N_{x-1} - N_{x+n-1}) = M_x - M_{x+n} + D_{x+n} + \pi(R_x - R_{x+n} - n\,M_{x+n})$$

$$\pi[N_{x-1} - N_{x+n-1} - (R_x - R_{x+n} - n\,M_{x+n})] = M_x - M_{x+n} + D_{x+n}$$

$$\pi = \frac{M_x - M_{x+n} + D_{x+n}}{N_{x-1} - N_{x+n-1} - (R_x - R_{x+n} - n\,M_{x+n})}$$

Suppose an endowment insurance with return of all net annual premiums only upon survival. The value is $A^{1}_{x\overline{n|}} + (1 + n\,\pi)\,{}_{n}E_x$ and the premium is found, thus:

$$\pi = \frac{A^{1}_{x\overline{n|}} + (1 + n\,\pi)\,{}_{n}E_x}{1 + a_{x\,\overline{n-1|}}}$$

$$= \frac{M_x - M_{x+n} + (1 + n\,\pi)\,D_{x+n}}{D_x} \div \frac{N_{x-1} - N_{x+n-1}}{D_x}$$

$$= \frac{M_x - M_{x+n} + (1 + n\,\pi)\,D_{x+n}}{N_{x-1} - N_{x+n-1}}$$

$$\pi(N_{x-1} - N_{x+n-1}) = M_x - M_{x+n} + (1 + n\,\pi)\,D_{x+n}$$

$$\pi(N_{x-1} - N_{x+n-1} - n\,D_{x+n}) = M_x - M_{x+n} + D_{x+n}$$

$$\pi = \frac{M_x - M_{x+n} + D_{x+n}}{N_{x-1} - N_{x+n-1} - n\,D_{x+n}}$$

ENDOWMENTS AND INSURANCES WITH RETURN OF NET ANNUAL PREMIUMS, LIMITED PAYMENTS.

Suppose an insurance for life paid for by t annual premiums, with return of net annual premiums at death. This is composed of an insurance for life and an insurance of π, increased by π annually for t years, and thereafter constant. The value is, then, $A_x + \pi(I_{\overline{t|}}A)_x$ and the limited annual premium for t years is found, thus:

$$\pi = \frac{A_x + \pi(I_{\overline{t|}}A)_x}{1 + a_{x\,\overline{t-1|}}}$$

$$= \frac{M_x + \pi(R_x - R_{x+t})}{D_x} \div \frac{N_{x-1} - N_{x+\overline{t-1}}}{D_x}$$

$$= \frac{M_x + \pi(R_x - R_{x+t})}{N_{x-1} - N_{x+t-1}}$$

$$\pi(N_{x-1} - N_{x+t-1}) = M_x + \pi(R_x - R_{x+t})$$

$$\pi[N_{x-1} - N_{x+t-1} - (R_x - R_{x+t})] = M_x$$

$$\pi = \frac{M_x}{N_{x-1} - N_{x+t-1} - (R_x - R_{x+t})}$$

Suppose a like insurance except that return of premiums is only in case death occurs during premium-paying period. The value will then be $A_x + \pi(IA)^1_{x\,\overline{t|}}$ and the limited annual premium for t years will be found, thus:

$$\pi = \frac{A_x + \pi(IA)^1_{x\,\overline{t|}}}{1 + a_{x\,\overline{t-1|}}}$$

$$= \frac{M_x + \pi(R_x - R_{x+t} - t\,M_{x+t})}{D_x} \div \frac{N_{x-1} - N_{x+\overline{t-1}}}{D_x}$$

$$= \frac{M_x + \pi(R_x - R_{x+t} - t\,M_{x+t})}{N_{x-1} - N_{x+t-1}}$$

$$\pi(N_{x-1} - N_{x+t-1}) = M_x + \pi(R_x - R_{x+t} - t\,M_{x+t})$$

$$\pi[N_{x-1} - N_{x+t-1} - (R_x - R_{x+t} - t\,M_{x+t})] = M_x$$

$$\pi = \frac{M_x}{N_{x-1} - N_{x+t-1} - (R_x - R_{x+t} - t\,M_{x+t})}$$

Suppose a pure endowment in n years, paid for by t premiums, with return of net premiums in event of death during the endowment period. The value will then be ${}_nE_x$ + $\pi(I\,A)^{1}_{x\,\overline{t|}} + t\,\pi\ {}_{t|}A^{1}_{x\,\overline{n-t|}}$ and the limited annual premium for t years may be found, thus:

$$\pi = \frac{{}_nE_x + \pi(I\,A)^{1}_{x\,\overline{t|}} + t\,\pi\ {}_{t|}A^{1}_{x\,\overline{n-t|}}}{1 + a_{x\,\overline{t-1|}}}$$

$$= \frac{D_{x+n} + \pi(R_x - R_{x+t} - t\,M_{x+t}) + t\,\pi\,(M_{x+t} - M_{x+n})}{D_x}$$

$$\div \frac{N_{x-1} - N_{x+t-1}}{D_x}$$

$$= \frac{D_{x+n} + \pi(R_x - R_{x+t} - t\,M_{x+t}) + t\,\pi\,(M_{x+t} - M_{x+n})}{N_{x-1} - N_{x+t-1}}$$

$\pi\,(N_{x-1} - N_{x+t-1}) = D_{x+n} + \pi(R_x - R_{x+t} - t\,M_{x+t}) + t\,\pi\,(M_{x+t} - M_{x+n})$

$\pi\,[N_{x-1} - N_{x+t-1} - (R_x - R_{x+t} - t\,M_{x+t}) - t\,(M_{x+t} - M_{x+n})\,] = D_{x+n}$

$$\pi = \frac{D_{x+n}}{N_{x-1} - N_{x+t-1} - (R_x - R_{x+t} - t\,M_{x+t}) - t\,(M_{x+t} - M_{x+n})}$$

Suppose an endowment insurance for n years paid for by t premiums, with return of premiums in event of death during the endowment period or in event of survival. The value is $(1 + t\,\pi)\ {}_nE_x + {}_{|n}A_x + \pi(I\,A^{1}_{x\,\overline{t|}}) + t\,\pi\ {}_{t|}A^{1}_{x\,\overline{n-t|}}$ and the limited annual premium for t years may be found, thus:

$$\pi = \frac{(1 + t\,\pi)\ {}_nE_x + {}_{|n}A_x + \pi(I\,A^{1}_{x\,\overline{t|}}) + t\,\pi\ {}_{t|}A^{1}_{x\,\overline{n-t|}}}{1 + a_{x\,\overline{t-1|}}}$$

$$= \left[\frac{(1 + t\,\pi)\,D_{x+n} + M_x - M_{x+n} + \pi(R_x - R_{x+t} - t\,M_{x+t})}{D_x} + \frac{t\,\pi\,(M_{x+t} - M_{x+n})}{D_x}\right] \div \frac{N_{x-1} - N_{x+t-1}}{D_x}$$

$$= \frac{(1 + t\pi)\, D_{x+n} + M_x - M_{x+n} + \pi\,(R_x - R_{x+t} - t\, M_{x+t}) + t\,\pi\,(M_{x+t} - M_{x+n})}{N_{x-1} - N_{x+t-1}}$$

$$\pi(N_{x-1} - N_{x+t-1}) = (1 + t\,\pi)\, D_{x+n} + M_x - M_{x+n} + \pi\,(R_x - R_{x+t} - t\, M_{x+t}) + t\,\pi\,(M_{x+t} - M_{x+n})$$

$$\pi[N_{x-1} - N_{x+t-1} - t\, D_{x+n} - (R_x - R_{x+t} - t\, M_{x+t}) - t\,(M_{x+t} - M_{x+n})\,] = M_x - M_{x+n} + D_{x+n}$$

$$\pi = \frac{M_x - M_{x+n} + D_{x+n}}{N_{x-1} - N_{x+t-1} - t\, D_{x+n} - (R_x - R_{x+t} - t\, M_{x+t}) - t\,(M_{x+t} - M_{x+n})}$$

Suppose an endowment insurance for n years paid for by t premiums, with return of net premiums only in event of death during endowment period. The value is $A_{x\overline{n|}} + \pi(I\,A)^1_{x\overline{t|}} + t\,\pi_t|A^1_{x\overline{n-t|}}$ and the annual premium may be found, thus:

$$\pi = \frac{A_{x\overline{n|}} + \pi(I\,A)^1_{x\overline{t|}} + t\,\pi_t|A^1_{x\overline{n-t|}}}{1 + a_{x\overline{t-1|}}}$$

$$= \frac{\dfrac{M_x - M_{x+n} + D_{x+n} + \pi\,(R_x - R_{x+t} - t\,M_{x+t}) + t\,\pi\,(M_{x+t} - M_{x+n})}{D_x}}{\dfrac{N_{x-1} - N_{x+t-1}}{D_x}}$$

$$= \frac{M_x - M_{x+n} + D_{x+n} + \pi\,(R_x - R_{x+t} - t\,M_{x+t}) + t\,\pi(M_{x+t} - M_{x+n})}{N_{x-1} - N_{x+t-1}}$$

$$\pi(N_{x-1} - N_{x+t-1}) = M_x - M_{x+n} + D_{x+n} + \pi(R_x - R_{x+t} - t\,M_{x+t}) + t\,\pi\,(M_{x+t} - M_{x+n})$$

$$\pi[N_{x-1} - N_{x+t-1} - (R_x - R_{x+t} - t\,M_{x+t}) - t\,(M_{x+t} - M_{x+n})] = M_x - M_{x+n} + D_{x+n}$$

$$\pi = \frac{M_x - M_{x+n} + D_{x+n}}{N_{x-1} - N_{x+t-1} - (R_x - R_{x+t} - t\,M_{x+t}) - t\,(M_{x+t} - M_{x+n})}$$

Suppose an endowment insurance for n years, paid for by t annual premiums to be returned only upon survival. The value is $(1 + t\,\pi)\,{}_nE_x + A^1_{x\overline{n|}}$ and the annual premium is found, thus:

$$\pi = \frac{(1 + t\,\pi)\ {}_{n}E_{x} + A^{1}_{x\,\overline{n|}}}{1 + a_{x\,\overline{t-1|}}}$$

$$= \frac{(1 + t\,\pi)\ D_{x+n} + M_{x} - M_{x+n}}{D_{x}} \div \frac{N_{x-1} - N_{x+t-1}}{D_{x}}$$

$$= \frac{(1 + t\,\pi)\ D_{x+n} + M_{x} - M_{x+n}}{N_{x-1} - N_{x+t-1}}$$

$$\pi\,(N_{x-1} - N_{x+t-1}) = (1 + t\,\pi)\ D_{x+n} + M_{x} - M_{x+n}$$

$$\pi\,(N_{x-1} - N_{x+t-1} - t\,D_{x+n}) = D_{x+n} + M_{x} - M_{x+n}$$

$$\pi = \frac{M_{x} - M_{x+n} + D_{x+n}}{N_{x-1} - N_{x+t-1} - t\,D_{x+n}}$$

ENDOWMENTS AND INSURANCES WITH RETURN OF GROSS ANNUAL PREMIUMS, CONTINUOUS THROUGHOUT TERM.

TAKING $P(1 + k) + c$ as our formula for the value of P', the gross annual premium, suppose an insurance for life with return of gross annual premiums. The value is plainly $A_x + [\pi(1 + k) + c]\,(I\,A)_x$ and the net annual premium is found thus:

$$\pi = \frac{A_x + [\pi(1 + k) + c]\,(I\,A)_x}{1 + a_x}$$

$$= \frac{M_x + [\pi(1 + k) + c]\,R_x}{D_x} \div \frac{N_{x-1}}{D_x}$$

$$= \frac{M_x + [\pi(1 + k) + c]\,R_x}{N_{x-1}}$$

$$\pi N_{x-1} = M_x + [\pi(1 + k) + c]\,R_x$$

$$\pi[N_{x-1} - (1 + k)\,R_x] = M_x + c\,R_x$$

$$\pi = \frac{M_x + c\,R_x}{N_{x-1} - (1 + k)\,R_x}$$

Suppose a pure endowment due in n years with return of all gross annual premiums if death occurs within n years. The value is ${}_nE_x + [\pi(1 + k) + c]\,(I\,A)^1_{x\,\overline{n|}}$ and the net annual premium is found, thus:

$$\pi = \frac{{}_nE_x + [\pi(1 + k) + c]\,(I\,A)^1_{x\,\overline{n|}}}{1 + a_{x\,\overline{n-1|}}}$$

$$= \frac{D_{x+n} + [\pi(1 + k) + c]\,(R_x - R_{x+n} - n\,M_{x+n})}{D_x} \div \frac{N_{x-1} - N_{x+n-1}}{D_x}$$

$$= \frac{D_{x+n} + [\pi(1 + k) + c]\,(R_x - R_{x+n} - n\,M_{x+n})}{N_{x-1} - N_{x+n-1}}$$

$$\pi(N_{x-1} - N_{x+n-1}) = D_{x+n} + [\pi(1 + k) + c]\,(R_x - R_{x+n} - n\,M_{x+n})$$

$$\pi[N_{x-1} - N_{x+n-1} - (1 + k)\,(R_x - R_{x+n} - n\,M_{x+n})] = D_{x+n} + c\,(R_x - R_{x+n} - n\,M_{x+n})$$

$$\pi = \frac{D_{x+n} + c\,(R_x - R_{x+n} - n\,M_{x+n})}{N_{x-1} - N_{x+n-1} - (1+k)\,(R_x - R_{x+n} - n\,M_{x+n})}$$

Suppose an insurance for n years with return of gross annual premiums. The value is $A^1_{x\overline{n|}} + [\pi(1 + k) + c]\,(IA)^1_{x\overline{n|}}$ and the net annual premium is found, thus:

$$\pi = \frac{A^1_{x\overline{n|}} + [\pi(1 + k) + c]\,(I\,A)^1_{x\overline{n|}}}{1 + a_{x\,\overline{n-1|}}}$$

$$= \frac{M_x - M_{x+n} + [\pi(1 + k) + c]\,(R_x - R_{x+n} - n\,M_{x+n})}{D_x} \div \frac{N_{x-1} - N_{x+n-1}}{D_x}$$

$$= \frac{M_x - M_{x+n} + [\pi(1 + k) + c]\,(R_x - R_{x+n} - n\,M_{x+n})}{N_{x-1} - N_{x+n-1}}$$

$$\pi(N_{x-1} - N_{x+n-1}) = M_x - M_{x+n} + [\pi(1 + k) + c]\,(R_x - R_{x+n} - n\,M_{x+n})$$

$$\pi[N_{x-1} - N_{x+n-1} - (1 + k)\,(R_x - R_{x+n} - n\,M_{x+n}) = M_x - M_{x+n} + c\,(R_x - R_{x+n} - n\,M_{x+n})$$

$$\pi = \frac{M_x - M_{x+n} + c\,(R_x - R_{x+n} - n\,M_{x+n})}{N_{x-1} - N_{x+n-1} - (1 + k)\,(R_x - R_{x+n} - n\,M_{x+n})}$$

Suppose an endowment insurance with return of all gross annual premiums either at death or maturity. The value is $[1 + n\,\pi\,(1 + k) + n\,c]\,{}_nE_x + A^1_{x\overline{n|}} + [\pi\,(1 + k) + c]\,(I\,A)^1_{x\overline{n|}}$ and the net annual premium is found, thus:

$$\pi = \frac{[1 + n\pi(1 + k) + n\,c]\,{}_nE_x + A^1_{x\overline{n|}} + [\pi(1 + k) + c]\,(I\,A)^1_{x\overline{n|}}}{1 + a_{x\,\overline{n-1|}}}$$

$$= \frac{[1 + n\pi(1 + k) + n\,c]\,D_{x+n} + M_x - M_{x+n} + [\pi(1 + k) + c]\,(R_x - R_{x+n} - n\,M_{x+n})}{D_x} \div \frac{N_{x-1} - N_{x+n-1}}{D_x}$$

$$= \frac{[1 + n\,\pi(1 + k) + n\,c]\,D_{x+n} + M_x - M_{x+n} + [\pi(1 + k) + c]\,(R_x - R_{x+n} - n\,M_{x+n})}{N_{x-1} - N_{x+n-1}}$$

$$\pi(N_{x-1} - N_{x+n-1}) = [1 + n\pi(1+k) + nc]\, D_{x+n} + M_x - M_{x+n} + [\pi(1+k) + c]\,(R_x - R_{x+n} - n\,M_{x+n})$$

$$\pi[N_{x-1} - N_{x+n-1} - (1+k)(n\,D_{x+n}) - (1+k)(R_x - R_{x+n} - n\,M_{x+n})] = (1 + nc)\, D_{x+n} + M_x - M_{x+n} + c\,(R_x - R_{x+n} - n\,M_{x+n})$$

$$\pi = \frac{(1 + nc)\, D_{x+n} + M_x - M_{x+n} + c\,(R_x - R_{x+n} - n\,M_{x+n})}{N_{x-1} - N_{x+n-1} - (1+k)(n\,D_{x+n}) - (1+k)(R_x - R_{x+n} - n\,M_{x+n})}$$

Suppose an endowment insurance for n years with all gross premiums returned only in case of death. The value is $A_{x\overline{n|}} + [\pi(1+k) + c]\,(I\,A)^1_{x\overline{n|}}$ and the net annual premium is found, thus:

$$\pi = \frac{A_{x\overline{n|}} + [\pi(1+k) + c]\,(I\,A)^1_{x\overline{n|}}}{1 + a_{x\overline{n-1|}}}$$

$$= \frac{\dfrac{M_x - M_{x+n} + D_{x+n} + [\pi(1+k) + c]\,(R_x - R_{x+n} - n\,M_{x+n})}{D_x}}{\dfrac{N_{x-1} - N_{x+n-1}}{D_x}}$$

$$= \frac{M_x - M_{x+n} + D_{x+n} + [\pi(1+k) + c]\,(R_x - R_{x+n} - n\,M_{x+n})}{N_{x-1} - N_{x+n-1}}$$

$$\pi(N_{x-1} - N_{x+n-1}) = M_x - M_{x+n} + D_{x+n} + [\pi(1+k) + c]\,(R_x - R_{x+n} - n\,M_{x+n})$$

$$\pi[N_{x-1} - N_{x+n-1} - (1+k)(R_x - R_{x+n} - n\,M_{x+n})] = M_x - M_{x+n} + D_{x+n} + c\,(R_x - R_{x+n} - n\,M_{x+n})$$

$$\pi = \frac{M_x - M_{x+n} + D_{x+n} + c\,(R_x - R_{x+n} - n\,M_{x+n})}{N_{x-1} - N_{x+n-1} - (1+k)(R_x - R_{x+n} - n\,M_{x+n})}$$

Suppose an endowment insurance with return of all gross annual premiums only upon survival. The value is $A^1_{x\overline{n|}} + [1 + n\pi(1+k) + n\,c]\,{}_nE_x$ and the net annual premium is found, thus:

$$\pi = \frac{A^1_{x\overline{n|}} + [1 + n\pi(1+k) + n\,c]\,{}_nE_x}{1 + a_{x\overline{n-1|}}}$$

$$=\frac{\dfrac{M_x - M_{x+n}+[1 + n\,\pi(1 + k) + n\,c]\,D_{x+n}}{D_x}}{\dfrac{N_{x-1} - N_{x+n-1}}{D_x}}$$

$$=\frac{M_x - M_{x+n}+ [1 + n\,\pi(1 + k) + n\,c]\,D_{x+n}}{N_{x-1} - N_{x+n-1}}$$

$$\pi\,(N_{x-1} - N_{x+n-1}) = M_x - M_{x+n}+ [1 + n\,\pi\,(1 + k) + n\,c]\,D_{x+n}$$

$$\pi\,[N_{x-1} - N_{x+n-1} - n\,(1 + k)\,D_{x+n}] = M_x - M_{x+n}+ (1 + n\,c)\,D_{x+n}$$

$$\pi=\frac{M_x - M_{x+n}+ (1 + n\,c)\,D_{x+n}}{N_{x-1} - N_{x+n-1} - n\,(1 + k)\,D_{x+n}}$$

ENDOWMENTS AND INSURANCES WITH RETURN OF GROSS ANNUAL PREMIUMS, LIMITED PAYMENTS.

Suppose an insurance for life paid for by t annual premiums, with return of gross annual premiums at death. The value is $A_x + [\pi(1 + k) + c]\,(I_{\overline{t|}}\,A)_x$ and the annual premium is found, thus:

$$\pi = \frac{A_x + [\pi(1 + k) + c]\,(I_{\overline{t|}}\,A)_x}{1 + a_{x\,\overline{t-1|}}}$$

$$= \frac{M_x + [\pi(1 + k) + c]\,(R_x - R_{x+t})}{D_x} \div \frac{N_{x-1} - N_{x+t-1}}{D_x}$$

$$= \frac{M_x + [\pi(1 + k) + c]\,(R_x - R_{x+t})}{N_{x-1} - N_{x+t-1}}$$

$$\pi(N_{x-1} - N_{x+t-1}) = M_x + [\pi(1 + k) + c]\,(R_x - R_{x+t})$$

$$\pi[N_{x-1} - N_{x+t-1} - (1 + k)\,(R_x - R_{x+t})] = M_x + c\,(R_x - R_{x+t})$$

$$\pi = \frac{M_x + c\,(R_x - R_{x+t})}{N_{x-1} - N_{x+t-1} - (1 + k)\,(R_x - R_{x+t})}$$

Suppose a like insurance except that return of gross annual premiums is only in case death occurs during premium-paying period. The value is $A_x + [\pi(1 + k) + c]$ $(I\,A)^1_{x\,\overline{t|}}$ and the net annual premium is found, thus:

$$\pi = \frac{A_x + [\pi(1 + k) + c]\,(I\,A)^1_{x\,\overline{t|}}}{1 + a_{x\,\overline{t-1|}}}$$

$$= \frac{\dfrac{M_x + [\pi(1 + k) + c]\,(R_x - R_{x+t} - tM_{x+t})}{D_x}}{\dfrac{N_{x-1} - N_{x+t-1}}{D_x}}$$

$$= \frac{M_x + [\pi(1 + k) + c]\,(R_x - R_{x+t} - t\,M_{x+t})}{N_{x-1} - N_{x+t-1}}$$

$$\pi(N_{x-1} - N_{x+t-1}) = M_x + [\pi(1 + k) + c]\,(R_x - R_{x+t} - t\,M_{x+t})$$

$$\pi[N_{x-1} - N_{x+t-1} - (1+k)(R_x - R_{x+t} - t\,M_{x+t})] = M_x + c\,(R_x - R_{x+t} - t\,M_{x+t})$$

$$\pi = \frac{M_x + c\,(R_x - R_{x+t} - t\,M_{x+t})}{N_{x-1} - N_{x+t-1} - (1+k)(R_x - R_{x+t} - t\,M_{x+t})}$$

Suppose a pure endowment in n years, paid for by t premiums, with return of gross premiums in event of death during the endowment period. The value is ${}_nE_x + [\pi(1+k)+c]\,(I\,A)^1_{x\overline{t|}} + t\,[\pi(1+k)+c]\,{}_{t|}A^1_{x\overline{n-t|}}$ and the net annual premium may be found, thus:

$$\pi = \frac{{}_nE_x + [\pi(1+k)+c]\,(I\,A)^1_{x\overline{t|}} + t\,[\pi(1+k)+c]\,{}_{t|}A^1_{x\overline{n-t|}}}{1 + a_{x\overline{t-1|}}}$$

$$= \frac{\dfrac{D_{x+n} + [\pi(1+k)+c]\,(R_x - R_{x+t} - t\,M_{x+t}) + t[\pi(1+k)+c]\,(M_{x+t} - M_{x+n})}{D_x}}{\dfrac{N_{x-1} - N_{x+t-1}}{D_x}}$$

$$= \frac{D_{x+n} + [\pi(1+k)+c]\,(R_x - R_{x+t} - t\,M_{x+t}) + t\,[\pi(1+k)+c]\,(M_{x+t} - M_{x+n})}{N_{x-1} - N_{x+t-1}}$$

$$\pi(N_{x-1} - N_{x+t-1}) = D_{x+n} + [\pi(1+k)+c]\,(R_x - R_{x+t} - t\,M_{x+t}) + t\,[\pi(1+k)+c]\,(M_{x+t} - M_{x+n})$$

$$\pi[N_{x-1} - N_{x+t-1} - (1+k)(R_x - R_{x+t} - t\,M_{x+t}) - t\,(1+k)(M_{x+t} - M_{x+n})$$

$$= D_{x+n} + c\,(R_x - R_{x+t} - t\,M_{x+t}) + t\,c\,(M_{x+t} - M_{x+n})$$

$$\pi = \frac{D_{x+n} + c\,(R_x - R_{x+t} - t\,M_{x+t}) + t\,c\,(M_{x+t} - M_{x+n})}{N_{x-1} - N_{x+t-1} - (1+k)(R_x - R_{x+t} - t\,M_{x+t}) - t\,(1+k)(M_{x+t} - M_{x+n})}$$

Suppose an endowment insurance for n years paid for by t premiums, with return of premiums either in event of death during the endowment period or of survival. The value is $A_{x\overline{n|}} + [t\,\pi\,(1+k) + t\,c]\,{}_nE_x + [\pi(1+k)+c]\,(I\,A)^1_{x\overline{t|}} + [t\,\pi\,(1+k) + t\,c]\,{}_{t|}{}_{n-t}A_x$ and the annual premium is found, thus:

$$\pi = \frac{A_{x\overline{n|}} + [t\pi(1+k)+tc]\,{}_nE_x + [\pi(1+k)+c]\,(IA)^1_{x\overline{t|}} + [t\pi(1+k)+tc]\,{}_{t|n-t}A_x}{1 + a_{x\overline{t-1|}}}$$

$$= \frac{\begin{matrix} M_x - M_{x+n} + D_{x+n}[1+t\pi(1+k)+tc] \\ + [\pi(1+k)+c](R_x - R_{x+t} - t\,M_{x+t}) + [t\pi(1+k)+tc] \\ (M_{x+t} - M_{x+n}) \end{matrix}}{D_x}$$

$$\div \frac{N_{x-1} - N_{x+t-1}}{D_x}$$

$$= \frac{\begin{matrix} M_x - M_{x+n} + D_{x+n}[1+t\pi(1+k)+tc] + [\pi(1+k)+c] \\ (R_x - R_{x+t} - t\,M_{x+t}) + [t\pi(1+k)+tc](M_{x+t} - M_{x+n}) \end{matrix}}{N_{x-1} - N_{x+t-1}}$$

Whence

$$\begin{aligned} \pi(N_{x-1} - N_{x+t-1}) = {} & M_x - M_{x+n} + D_{x+n}[1+t\pi(1+k)+tc] \\ & + [\pi(1+k)+c](R_x - R_{x+t} - t\,M_{x+t}) \\ & + [t\pi(1+k)+tc](M_{x+t} - M_{x+n}) \end{aligned}$$

$$\begin{aligned} \pi\{ (N_{x-1} - N_{x+t-1} & - t(1+k)D_{x+n} - (1+k)(R_x - R_{x+t} \\ & - t\,M_{x+t}) - t(1+k)(M_{x+t} - M_{x+n}) \} \\ = M_x - M_{x+n} & + (1+tc)D_{x+n} + c(R_x - R_{x+t} - t\,M_{x+t}) \\ & + tc(M_{x+t} - M_{x+n}) \end{aligned}$$

Whence

$$\pi = \frac{\begin{matrix} M_x - M_{x+n} + D_{x+n} + c(R_x - R_{x+t} - t\,M_{x+t}) + tc \\ (M_{x+t} - M_{x+n} + D_{x+n}) \end{matrix}}{\begin{matrix} N_{x-1} - N_{x+t-1} - t(1+k)(M_{x+t} - M_{x+n} + D_{x+n}) - \\ (1+k)(R_x - R_{x+t} - t\,M_{x+t}) \end{matrix}}$$

Suppose an endowment insurance for n years paid for in t years with return of gross annual premiums only in event of survival. The value is $[1 + t\pi(1+k) + tc]\,{}_nE_x + A^1_{x\overline{n|}}$ and the net annual premium is found, thus:

$$\pi = \frac{[1 + t\pi(1+k) + tc]\,{}_nE_x + A^1_{x\overline{n|}}}{1 + a_{x\overline{t-1|}}}$$

$$= \frac{[1 + t\pi(1+k) + tc]\,D_{x+n} + M_x - M_{x+n}}{D_x} \div \frac{N_{x-1} - N_{x+t-1}}{D_x}$$

$$= \frac{[1 + t\pi(1+k) + tc]\,D_{x+n} + M_x - M_{x+n}}{N_{x-1} - N_{x+t-1}}$$

$$\pi(N_{x-1} - N_{x+t-1}) = [1 + t\pi(1+k) + tc]\,D_{x+n} + M_x - M_{x+n}$$

$$\pi[N_{x-1}-N_{x+t-1}-t\,(1+k)\,D_{x+n}]=(1+t\,c)\,D_{x+n}+M_x-M_{x+n}$$

$$\pi=\frac{M_x-M_{x+n}+(1+t\,c)\,D_{x+n}}{N_{x-1}-N_{x+t-1}-t\,(1+k)\,D_{x+n}}$$

Suppose an endowment insurance for n years paid for by t premiums with return of gross annual premiums only in event of death during endowment period. The value is $A_{x\,\overline{n|}}+[\pi(1+k)+c]\,(I\,A)^1_{x\,\overline{t|}}+[t\,\pi\,(1+k)+t\,c]\;{}_{t|}A^1_{x\,\overline{n-t|}}$ and the net annual premium is found, thus:

$$\pi=\frac{A_{x\,\overline{n|}}+[\pi(1+k)+c]\,(I\,A)^1_{x\,\overline{t|}}+[t\,\pi\,(1+k)+t\,c]\;{}_{t|}A^1_{x\,\overline{n-t|}}}{1+a_{x\,\overline{t-1|}}}$$

$$=\frac{\dfrac{M_x-M_{x+n}+D_{x+n}+[\pi(1+k)+c]\,(R_x-R_{x+t}-t\,M_{x+t})+[t\,\pi(1+k)+t\,c]\,(M_{x+t}-M_{x+n})}{D_x}}{\dfrac{N_{x-1}-N_{x+t-1}}{D_x}}$$

$$=\frac{M_x-M_{x+n}+D_{x+n}+[\pi(1+k)+c]\,(R_x-R_{x+t}-t\,M_{x+t})+[t\,\pi(1+k)+t\,c]\,(M_{x+t}-M_{x+n})}{N_{x-1}-N_{x+t-1}}$$

$$\pi(N_{x-1}-N_{x+t-1})=M_x-M_{x+n}+D_{x+n}+[\pi(1+k)+c]\,(R_x-R_{x+t}-t\,M_{x+t})+[t\,\pi\,(1+k)+t\,c]\,(M_{x+t}-M_{x+n})$$

$$\pi[N_{x-1}-N_{x+t-1}-(1+k)\,(R_x-R_{x+t}-t\,M_{x+t})-t\,(1+k)\,(M_{x+t}-M_{x+n})$$

$$=M_x-M_{x+n}+D_{x+n}+c\,(R_x-R_{x+t}-t\,M_{x+t})+t\,c\,(M_{x+t}-M_{x+n})$$

$$\pi=\frac{(M_x-M_{x+n}+D_{x+n}+c\,(R_x-R_{x+t}-t\,M_{x+t})+t\,c\,(M_{x+t}-M_{x+n})}{N_{x-1}-N_{x+t-1}-(1+k)\,(R_x-R_{x+t}-t\,M_{x+t})-t\,(1+k)\,(M_{x+t}-M_{x+n})}$$

Suppose an endowment insurance for n years paid for in t years, with return of gross annual premiums only in event of death during premium-paying period. The value is

$A_{x\overline{n|}} + [\pi(1 + k) + c]\,(I\,A)^{1}_{x\overline{t|}}$ and the annual premium is found, thus:

$$\pi = \frac{A_{x\overline{n|}} + [\pi(1 + k) + c]\,(I\,A)^{1}_{x\overline{t|}}}{1 + a_{x\overline{t-1|}}}$$

$$= \frac{\dfrac{M_x - M_{x+n} + D_{x+n} + [\pi(1 + k) + c]\,(R_x - R_{x+t} - t\,M_{x+t})}{D_x}}{\dfrac{N_{x-1} - N_{x+t-1}}{D_x}}$$

$$= \frac{M_x - M_{x+n} + D_{x+n} + [\pi(1 + k) + c]\,(R_x - R_{x+t} - t\,M_{x+t})}{N_{x-1} - N_{x+t-1}}$$

$$\pi(N_{x-1} - N_{x+t-1}) = M_x - M_{x+n} + D_{x+n} + [\pi(1 + k) + c]\,(R_x - R_{x+t} - t\,M_{x+t})$$

$$\pi\,[N_{x-1} - N_{x+t-1} - (1 + k)\,(R_x - R_{x+t} - t\,M_{x+t})\,] = M_x - M_{x+n} + D_{x+n} + c\,(R_x - R_{x+t} - t\,M_{x+t})$$

$$\pi = \frac{M_x - M_{x+n} + D_{x+n} + c\,(R_x - R_{x+t} - t\,M_{x+t})}{N_{x-1} - N_{x+t-1} - (1 + k)\,(R_x - R_{x+t} - t\,M_{x+t})}$$

NET EXTRA PREMIUM FOR RETURN OF PREMIUMS.

The usual form of return premium policies in the United States is with return of all or of part of the premiums in event of death during a dividend or payment period of n years.

Two cases may be distinguished, viz.: When premiums are payable during the whole period of n years and when premiums are payable for but part of such period.

As to the first case, the net annual premium for an increasing insurance of 1 is found by the following formula:

$$\pi = \frac{(IA)_{x\,\overline{n}|}}{1 + a_{x\,\overline{n-1}|}} = \frac{R_x - R_{x+n} - n\,M_{x+n}}{N_{x-1} - N_{x+n-1}}.$$

As to the second case, let m = the payment term; and, for an insurance of 1 increasing by 1 each year for m — 1 years and then stationary at m for n — m years longer, we have:

$$\pi = \frac{R_x - R_{x+m} - m\,M_{x+n}}{N_{x-1} - N_{x+m-1}}.$$

If the original annual premium P′ were all that was to be returned at death, the net extra premium would be $\pi P'$ in all cases of full premium return; and it could be loaded as desired. For return of $\frac{1}{b}$ of the annual premium, the net extra premium would be $\frac{\pi}{b} P'$.

But usually it is desired to provide for the return of the extra premium as well as the original annual premium. First suppose, then, that the net extra premium is to be returned and let π have the same meaning as in the foregoing, while ϕ signifies the complete net extra premium and P'', the complete gross premium, with the extra premium included. Then we have:

$$P'' = P' + \phi = P' + \pi (P' + \phi).$$
$$\phi = \pi (P' + \phi).$$
$$\phi - \phi\pi = P'\pi.$$
$$\phi (1 - \pi) = P'\pi.$$
$$\phi = P' \frac{\pi}{1-\pi} \therefore P'' = P'\left(1 + \frac{\pi}{1-\pi}\right) = P'\left(\frac{1}{1-\pi}\right)$$

To find the net extra premium on the same conditions, when only $\frac{1}{b}$ times the full premium, including the net extra, is to be returned at death:

$$P'' = P' + \phi = P' + \frac{\pi}{b}\left(P' + \phi\right)$$
$$\phi = \frac{\pi}{b}(P' + \phi)$$
$$\phi\left(1 - \frac{\pi}{b}\right) = P'\frac{\pi}{b}$$
$$\phi = P'\frac{\frac{\pi}{b}}{1 - \frac{\pi}{b}}, \therefore P'' = \left(\frac{1}{1 - \frac{\pi}{b}}\right)$$

To find the extra premium for the return of the entire premium, including the extra premium loaded k per cent of itself:

$$P'' = P' + \phi' = P' + \pi (1 + k)(P' + \phi')$$
$$\phi' = \pi (1 + k)(P' + \phi')$$
$$\phi' [1 - \pi (1 + k)] = P' \pi (1 + k).$$
$$\phi' = P' \frac{\pi (1+k)}{1 - \pi (1+k)} \therefore P'' = P'\left(\frac{1}{1 - \pi (1+k)}\right)$$

The total premium to be returned is P'', which consists of the gross ordinary premium P' and the gross extra ϕ'. The net extra premium would be $\phi = \frac{\phi'}{1+k}$.

AGE AND PREMIUM KNOWN, TO APPROXIMATE THE TERM.

GIVEN the age x, the amount of the insurance and the net single premium $|_{n'}A_x$, to find the duration of the insurance, n′.

We have the formula:

$$|_{n'}A_x = \frac{M_x - M_{x+n'}}{D_x}$$

In this equation, the first term is a known quantity and both M_x and D_x are determinable quantities by mere reference to the commutation tables, the age x being known. This leaves but one quantity, $M_{x+n'}$ neither known nor determinable by inspection.

We find its value in terms of the known quantities, thus:

$$D_x\,(|_{n'}A_x) = M_x - M_{x+n'}$$
$$M_{x+n'} = M_x - D_x\,(|_{n'}A_x)$$

If it be desired to approximate the term in even years only, then by reference to the M column that value of M is found which most nearly corresponds to the ascertained value. This value determines the age (approximately x + n′) when the insurance ceases; the term is that age less the age at entry (x + n′— x = n′).

If it be desired to approximate more closely, then by reference to the M column find that value of M which is next less than the ascertained value. Giving this the symbol M_{x+n}, we have x + n — x = n or the number of entire years which the insurance will run. To find approximately the number of days additional, first ascertain the difference between M_{x+n} and M_{x+n+1}, and, second, the difference between M_{x+n} and $M_{x+n'}$. Now if these differences were precisely the same, $M_{x+n'}$ would be equal to M_{x+n+1} and n′

would be equal to n + 1; but $M_{x+n} - M_{x+n'} < M_{x+n} - M_{x+n+1}$ since M_{x+n} is the value of M in the table, next less than the ascertained value $M_{x+n'}$ and M_{x+n+1} the value next greater. But since if the difference were equal, the value of $M_{x+n'}$ would be equal to M_{x+n+1}, we may assume for approximation that if $M_{x+n} - M_{x+n'} = \frac{M_{x+n} - M_{x+n+1}}{2}$, n′ will be equal to n + ½ years; or let $M_{x+n} - M_{x+n'}$ be any fraction of $M_{x+n} - M_{x+n+1}$, as w, then $n' = n + w$.

This may, in order to approximate the excess over n years by days, be stated in the form of a proportion, thus:

$$M_{x+n} - M_{x+n+1} : M_{x+n} - M_{x+n'} :: 365 : n' - n$$

$$n' - n = \frac{365\,(M_{x+n} - M_{x+n'})}{M_{x+n} - M_{x+n+1}} = w.$$

The theory upon which this approximation is based is that of uniform deaths throughout the year, while strict adherence to fact calls for increasing mortality; but in this formula the error is on the safe side, the assumption being that the premium will be expended a little sooner than the mortality calls for. The approximation, however, closely corresponds to the practical system of apportioning mortuary cost, commonly employed by American companies.

In the case of annual premiums, the formulas are derived as follows:

$$|_{n'}P_x = \frac{M_x - M_{x+n'}}{N_{x-1} - N_{x+n'-1}}$$

In this equation all terms are known or determinable except $M_{x+n'}$ and $N_{x+n'-1}$. We may find the combined value of these thus:

$$|_{n'}P_x\,(N_{x-1} - N_{x+n'-1}) = M_x - M_{x+n'}$$
$$M_{x+n'} - |_{n'}P_x\,(N_{x+n'-1}) = M_x - |_{n'}P_x\,N_{x-1}$$

Now find the value of $M_{x+n} - |_{n'}P_x\,(N_{x+n-1})$, taking for x + n any age (near the age to which the insurance seems from inspection likely to run), as a trial value. Continue the trials until two values are found, next under and next over the ascertained value.

If an approximation in years only is desired, take the value most closely approximating the ascertained value, the exponent will be the age (x + n′) approximately when the insurance ceases; and the term may be found by deducting the age at entry, thus: $x + n' - x = n'$.

If an approximation in days is desired, then having found the values, next over and next under the ascertained values, let them be designated by the symbols, X_{x+n} and X_{x+n+1}, and let the ascertained value be designated by the symbol $X_{x+n'}$. Then since $X_{x+n} - X_{x+n'} < X_{x+n} - X_{x+n+1}$
and since if

$$X_{x+n} - X_{x+n'} = X_{x+n} - X_{x+n+1}$$
$$n' = n + 1,$$

we may assume that when $X_{x+n} - X_{x+n'} = \frac{X_{x+n} - X_{x+n+1}}{2}$,

$$n' = n + \tfrac{1}{2}$$

and, generally, when $X_{x+n} - X_{x+n'} = w\ (X_{x+n} - X_{x+n+1})$,
(w being any fraction)

$$n' = n + w$$

Or to approximate the excess over n years by days, we may employ the proportion:

$$X_{x+n} - X_{x+n+1}, : X_{x+n} - X_{x+n'} : : 365 : n' - n$$

$$n' - n = \frac{365\ (X_{x+n} - X_{x+n'})}{X_{x+n} - X_{x+n+1}} = w.$$

Given the age x, the amount of insurance and the net single premium, $A_{x\,\overline{n}|}$, of an endowment insurance, to find the term to maturity, n′.

We have the formula:

$$A_{x\,\overline{n}|} = \frac{M_x - M_{x-n'} + D_{x+n'}}{D_x}$$

In this equation, the known or determinable quantities are $A_{x\,\overline{n'}|}$, M_x and D_x and the unknown are $M_{x+n'}$ and $D_{x+n'}$.

We may get all the known quantities on one side and the unknown on another, thus:

$$D_x\,A_{x\,\overline{n'}|} = M_x - M_{x+n'} + D_{x+n'}$$

$$M_{x+n'} - D_{x+n'} = M_x - D_x\,A_{x\,\overline{n'}|}$$

Now find the value of $M_{x+n'} - D_{x+n'}$, taking for x + n′ any age x + n (near the age at which from inspection the endowment seems likely to mature) as a trial value and repeat until the values next under and next over the ascertained value have been found.

If an approximation in years only is desired, the age of the value most nearly equal to the ascertained value may be taken as the age at which the endowment will mature; and the term may be found by deducting the age at entry, thus $x + n' - x = n'$.

If an approximation in days is desired, then designate the values over and under the ascertained value by the symbols X_{x+n} and X_{x+n+1} and the ascertained value by $X_{x+n'}$ and proceed as the foregoing calculations where the same symbols were employed. The final formulas will be as before

$$n' = n + w$$

$$\text{and } n' - n = \frac{365\,(X_{x+n} - X_{x+n'})}{X_{x+n} - X_{x+n+1}}$$

In the case of annual premiums, the formulas may be derived as follows:

$$P_{x\,\overline{n'}|} = \frac{M_x - M_{x+n'} + D_{x+n'}}{N_{x-1} - N_{x+n'-1}}$$

In this equation $P_{x\,\overline{n'}|}$, M_x and N_{x-1} are known as determinable quantities and $M_{x+n'}$, $D_{x+n'}$ and $N_{x+n'-1}$ are unknown quantities. We may get the latter upon one side of the equation, thus:

$$P_{x\,\overline{n'}|}\,(N_{x-1} - N_{x+n'-1}) = M_x - M_{x+n'} + D_{x+n'}$$
$$M_{x+n'} - D_{x+n'} - P_{x\,\overline{n'}|}\,(N_{x+n'-1}) = M_x - P_{x\,\overline{n'}|}\,N_{x-1}$$

Proceed as before to find values, X_{x+n} and X_{x+n-1}, next under and over the ascertained values, $X_{x+n'}$ by which symbol $M_{x+n'} - D_{x+n'} - P_{x\,\overline{n'}|}\,(N_{x+n'-1})$ may be designated.

The age ($x+n$ or $x+n+1$) of the value X approximating $X_{x+n'}$ most nearly, will be the approximate age at maturity in even years and the term to maturity is $x+n' - x = n'$.

The approximations in fractions of a year and in years and days may be found by developing the calculation as before, resulting in the final formulas:

$$n' = n + w$$
$$\text{and } n' - n = \frac{365\,(X_{x+n} - X_{x+n'})}{X_{x+n} - X_{x+n+1}} = w.$$

Given the age x, the amount of insurance and the limited premium ${}_tP^1_{x\,\overline{n'}|}$ for t years, to find the term, n'.

We have the formula:

$${}_tP^1_{x\,\overline{n'}|} = \frac{M_x - M_{x+n'}}{N_{x-1} - N_{x+t-1}}$$

In this equation, the known or determinable quantities are ${}_tP^1_{x\,\overline{n'}|}$, M_x, N_{x-1} and N_{x+t-1} leaving only $M_{x+n'}$ as the unknown quantity. The equation takes the form:

$${}_tP^1_{x\,\overline{n'}|}\,(N_{x-1} - N_{x+t-1}) = M_x - M_{x+n'}$$
$$M_{x+n'} = M_x - {}_tP^1_{x\,\overline{n'}|}\,(N_{x-1} - N_{x+t-1})$$

Find by inspection of the M column, that value of M nearest the ascertained value and the age will approximate in even years the age at which the insurance will cease; the term is then found by deducting the age at entry, thus: $x + n' - x = n'$.

To approximate in years and fractions of a year or in years and days, find by inspection values, M_{x+n} and M_{x+n+1}, next under and over the ascertained value. Then proceed precisely as in finding the approximate term, the net single premium being given, reaching the formulas:

$$n' = n + w$$

$$\text{and } n' - n = \frac{365\,(M_{x+n} - M_{x+n'})}{M_{x+n} - M_{x+n+1}} = w.$$

Given the age x, the limited premium ${}_tP_{x\,\overline{n'|}}$ for t years and the amount of endowment insurance, to find the term in which the endowment matures, n'.

We have the formula:

$${}_tP_{x\,\overline{n'|}} = \frac{M_x - M_{x+n'} + D_{x+n'}}{N_{x-1} - N_{x+t-1}}$$

In this equation, the known or determinable quantities are ${}_tP_{x\,\overline{n'|}}$, M_x, N_{x-1} and N_{x+t-1}, and the unknown quantities $M_{x+n'}$ and $D_{x+n'}$. Getting the known quantities on one side of the equation, we have:

$${}_tP_{x\,\overline{n'|}}\,(N_{x-1} - N_{x+t-1}) = M_x - M_{x+n'} + D_{x+n'}$$

$$M_{x+n'} - D_{x+n'} = M_x - {}_tP_{x\,\overline{n'|}}\,(N_{x-1} - N_{x+t-1})$$

Find the value of $M_{x+n} - D_{x+n}$ from the commutation tables, taking for x + n any age (near the age at which from inspection the endowment seems likely to mature) as a trial value. Repeat until two values are found, next under and next over the ascertained value. These we may designate X_{x+n} and X_{x+n+1}, and the ascertained value $X_{x+n'}$. The same calculations already repeatedly employed will give the formulas:

$$n' = n + w$$

$$n' - n = \frac{365\,(X_{x+n} - X_{x+n'})}{X_{x+n} - X_{x+n+1}} = w.$$

Given the age x, the amount of the annuity and the net single premium $|_{n'}a_x$, to find the term for which annuity will continue.

We have the formula:

$$|_{n'}a_x = \frac{N_x - N_{x+n'}}{D_x}$$

In this equation, the known or determinable quantities are $|_{n'}a_x$, N_x and D_x and the unknown quantity $N_{x+n'}$. Clearing of fractions and separating out the unknown quantity, we get:

$$D_x\,(|_{n'}a_x) = N_x - N_{x+n'}$$

$$N_{x+n'} = N_x - D_x\,(|_{n}a_x)$$

Find the value N_{x+n} next under the ascertained values $N_{x+n'}$; x + n will then be the age attained when the last full annuity will be payable and x + n — x = n will be the term during which a full annuity is payable.

But a fractional annuity will be due one year later, on survival. If the difference $N_{x+n} - N_{x+n'}$ were equal to $N_{x+n} - N_{x+n+1}$, a whole annuity would be then payable. We may therefore approximate the fractional annuity by this proportion, letting w represent the fraction:

$$N_{x+n} - N_{x+n+1} : N_{x+n} - N_{x+n'} :: 1 : w$$

$$\therefore (N_{x+n} - N_{x+n+1})\,w = N_{x+n} - N_{x+n'}$$

$$w = \frac{N_{x+n} - N_{x+n'}}{N_{x+n} - N_{x+n+1}}$$

It follows then that the term of the annuity is n + 1 years, during n years of which a full annuity is payable and a fractional annuity, w, at the end of n + 1 years.

VALUATIONS, GROSS AND AGGREGATE.

THE liabilities of a life insurance company, aside from ascertaining immediate claims against it, consist of future death-claims, endowment and annuity payments. The company owes the whole amount insured by it, death being certain, subject to the continuance of the insurance in force; it must therefore be counted to owe the whole amount insured until some part of the insurance be discontinued.

The folly of diminishing this computed liability by estimating discontinuances is at once apparent when the case of insurances for whole life, to be paid for as currently enjoyed, is considered. Counting the company as liable for the whole amount insured and the insured as liable for all future premiums, liabilities and resources precisely balance as they should; discontinuances at any time remove also equivalent resources and liabilities. On the other hand, counting the company as liable for anything less than the whole amount insured because of estimated discontinuances will make the resources apparently greater from the start than the liabilities and will thus destroy the balance which should exist.

The company, then, is liable for all insurances for whole life that it assumes and for an ascertainable amount for all insurances for less than whole life. We may set out with a list of the insurances classified by ages and figure from our mortality table how many deaths will according to that table occur the first year, the second year, the third year—every year in fact until all lives have failed.

The only things that diminish the company's present liabilities for these death-claims are, first, that they are not now due and, second, that if it had the money in hand now to meet them, it could put it at interest until the respective claims were due. For these reasons, the actual present liabilities may be computed as less than the whole amount insured, but only by so much as equals the discount at a reasonable rate on the respective death claims and other payments from the dates when they will accrue to the present date. Thus we discount the claims due in one year by one year's interest, those due in two years by two years' interest, etc., summing all the discounted amounts to find the immediate liability.

The resources consist of two distinct classes, first, the actual cash and other assets and, second, the future premiums. These future premiums form a resource which will be realizable at least as rapidly as is requisite in order to meet death-claims. They may therefore be offset against the death-claims and other payments. This may be accomplished by ascertaining the present or discounted value of the premiums, by discounting sums due in one year by one year's interest, etc., and adding up the amounts thus arrived at.

It goes without saying that if the present value of the premiums plus the present assets is less than the present value of future death-claims and other payments, the company is insolvent.

From this it follows as a necessary corollary that the company in order to be solvent must have in reserve a sum at least equal to the difference between the value of all future death-claims and other payments which it must make and the value of all future premiums which it will receive.

Since the company is insolvent if it does not possess this reserve, it is clear that this reserve possesses the nature and has the effect of a liability instead of a surplus. The company may in its balance-sheet eliminate both the item of present value of future death-claims and the item of present value of future premiums and merely charge as a liability against present assets the reserve.

Originally balance-sheets exhibited as resources all present assets plus the present value of all future premiums and as liabilities all immediate claims plus the present value of all future claims. This style of balance-sheet is still generally employed in Great Britain, omitting the present assets, to bring out the reserve.

The shortened balance-sheet exhibits as resources only present assets and as liabilities immediate claims plus the reserve, which is the present value of future death-claims less the present value of future premiums.

By converting the first balance-sheet into an equation, we shall plainly see that the second is directly derived from it. We have:

Present assets + Present value premiums = Immediate claims + Present value future claims + surplus.

Present assets = Immediate claims + (Present value future claims — Present value premiums = Reserve) + Surplus.

We thus find the aggregate reserve which the company needs to preserve intact in order to meet its liabilities. In computing this reserve originally actuaries inclined to compute the present value of future premiums from the gross or office premiums, allowing nothing for expenses. The real assumption was that surplus enough would arise during the operation to cover expenses, which was a reasonable assumption once, whatever may be thought of it as applied to present conditions. This method is called "gross valuation," an apt name both because the "gross" premiums are dealt with and also because the reserve is regarded as one mass and the policies are dealt with in gross and not singly. Under this method, the addition of new policies to the insurances reduced instead of increased the net balance of liability, it often being years before the present value of its future premiums fell below the present value of its future costs.

The very mention of the influence and effect of the addition of new policies indicates that the aggregate reserve is composed of separate values of individual insurances to be ascertained by finding the differences between the probable future premiums and the probable future costs of each, the sum of which differences equals the aggregate reserves.

It was always foolish at best to count the difference a present resource when the value of future premiums exceeded the value of future costs for the reason that no enforceable obligation to pay future premiums exists. At most it is only safe to count future premiums as a resource in an amount equal to future costs.

When agency methods and expansion of business made expenses nearly equal to the entire loading which had been added to premiums to meet expenses and other contingent requirements, the gross system of valuation became unavailable. One of the first public acts of Elizur Wright was to make this clear. He proposed instead what is known as net valuation and he procured it to be adopted by the State of Massachusetts as a test of solvency.

The net system differs from the gross only in one particular and that is that in figuring resources only net premiums are considered, the equations becoming:

Present assets + Present value net premiums = Immediate claims + Present value future claims + Surplus.

Present assets = Immediate claims + (Present value future claims — Present value net premiums = Reserve) + Surplus.

The aggregate net reserve is composed of the net individual reserves, *i. e.*, of the sum of the differences of the discounted probable future costs and the discounted probable future premiums of each insurance.

NET VALUATIONS, ANNUITIES, INSURANCES AND ENDOWMENTS, PROSPECTIVE METHOD.

GIVEN a life annuity, a_x, to find its value after n years.

There still remain all the payments to be made thereon after age x + n has been passed.

But the value of an annuity from age x + n is a_{x+n}. It follows that the value of the annuity a_x after n years is a_{x+n}.

Given a life insurance for a single premium A_x, to find its value after n years.

There still remain all the insurance costs after age x + n.

The value of an insurance from age x + n, comprising all these costs, is A_{x+n}. It follows that the value of an insurance of 1 after n years is A_{x+n}.

Given a life insurance for annual premiums, P_x, to find its value after n years.

At the outset at age x, the value of the insurance benefits and the value of the premiums balance. For by construction

$$P_x (1 + a_x) = A_x,$$

the first term representing the value of the premium just received together with all future premiums and the second term the value of future insurance benefits.

But at the end of n years, the value of future insurance benefits has become A_{x+n} instead of A_x and the value of future premiums $P_x (1 + a_{x+n})$ instead of $P_x (1 + a_x)$. Moreover

$$A_{x+n} > A_x$$

$$\text{while } P_x (1 + a_{x+n}) < P_x (1 + a_x)$$

That is, the net value of the insurance increases with the lapse of years while the net value of the premiums to be received diminishes with the lapse of years.

The difference between the net value of the insurance (A_{x+n}) and the net value of the premiums to be received $P_x\,(1 + a_{x+n})$ is called the value or reserve of the policy and may be designated by the symbol ${}_nV$, followed by a symbol designating the form of policy.

We then have the equation:

$$A_{x+n} = {}_nV\,P_x + P_x\,(1 + a_{x+n})$$

$$\therefore {}_nV\,P_x = A_{x+n} - P_x\,(1 + a_{x+n})$$

$$= \frac{M_{x+n}}{D_{x+n}} - \frac{P_x\,N_{x+n-1}}{D_{x+n}} = \frac{M_{x+n} - P_x\,N_{x+n-1}}{D_{x+n}}$$

Which, in language, means that the value of an annual premium whole life insurance at the end of any policy year is equal to the net single premium at the age attained less the value of an immediate annuity equal to the annual premium. ${}_nV\,P_x$ is usually written ${}_nV_x$.

In conversion formulas for annuities and insurances we had these two:

$$A_{x+n} = 1 - (1 - v)\,(1 + a_{x+n})$$

$$\text{and } P_x = \frac{1}{1 + a_x} - (1 - v)$$

Substituting these equivalent values in the formula for ${}_nV_x$, we have

$${}_nV_x = 1 - (1 - v)\,(1+a_{x+n}) - \left[\frac{1}{1 + a_x} - (1 - v)\right](1 + a_{x+n})$$

$$= 1 - \left(\frac{1}{1 + a_x}\right)(1 + a_{x+n})$$

$$= 1 - \frac{1 + a_{x+n}}{1 + a_x} = \frac{a_x - a_{x+n}}{1 + a_x}$$

$$= 1 - \frac{D_x N_{x+n-1}}{D_{x+n} N_{x-1}}$$

The problem may also be solved another way, viz.: the insurance now, n years after issue at age x, is worth A_{x+n} or $P_{x+n}(1+a_{x+n})$; the future premiums on the contrary are worth but $P_x(1+a_{x+n})$ and the difference between the value of the insurance and the value of the premiums, ${}_nV_x$, is wanted.

We have the equations:

$$A_{x+n} = P_{x+n}(1+a_{x+n})$$

$$\text{and } A_{x+n} = {}_nV_x + P_x(1+a_{x+n})$$

$${}_nV_x + P_x(1+a_{x+n}) = P_{x+n}(1+a_{x+n})$$

$$\text{and } {}_nV_x = (P_{x+n} - P_x)(1+a_{x+n})$$

In language, this may be stated thus: the value or reserve of a life insurance with annual premiums at the end of any policy year is equal to the value of an immediate annuity for the difference between the annual premium at age of entry and the annual premium at age attained.

In the case of limited premiums, given the premium ${}_tP_x$, to find the value after n years.

At the outset the value of the insurance was A_x and the value of the premiums was ${}_tP_x(1+|_{t-1}a_x)$ and the equation stood thus:

$$A_x = {}_tP_x(1+|_{t-1}a_x)$$

Now the value of the insurance is A_{x+n} and the value of the premiums ${}_tP_x(1+|_{t-n-1}a_{x+n})$ only; and the equation becomes:

$$A_{x+n} = {}_nV\,{}_tP_x + {}_tP_x(1+|_{t-n-1}a_{x+n})$$

$${}_nV\,{}_tP_x = A_{x+n} - {}_tP_x(1+|_{t-n-1}a_{x+n})$$

In the case of a temporary insurance with single premium, given the single premium $|_nA_x$, to find the value after m years.

After m years there will remain an insurance beginning at age x + m and running n — m years, the value of which is $|_{n-m}A_{x+m}$.

In the case of a temporary insurance with annual premiums, given the premium $|_{n}P_x$, to find the value after m years.

At the outset the values of the insurance and of the premiums balanced thus:

$$|_{n}A_x = |_{n}P_x\,(1 + |_{n-1}\,a_x)$$

Now the value of the insurance stands at $|_{n-m}A_{x+m}$ and the value of the premiums at $|_{n}P_x\,(1 + |_{n-m-1}\,a_{x+m})$ and the equation stands:

$$|_{n-m}A_{x+m} = {}_{m}V|_{n}P_x + |_{n}P_x\,(1 + |_{n-m-1}\,a_{x+m})$$
$${}_{m}V|_{n}P_x = |_{n-m}\,A_{x+m} - |_{n}P_x\,(1 + |_{n-m-1}\,a_{x+m})$$

In the case of an endowment insurance with single premium, given the premium $A_{x\,\overline{n}|}$, to find the value in m years.

After m years there will remain n — m years insurance and an endowment due in n — m years; in other words, the value will be that of an endowment insurance for n — m years entered into at age x + m, thus $A_{x+m\,\overline{n-m}|}$.

In the case of an endowment insurance with annual premium, given the premium $P_{x\,\overline{n}|}$, to find the value in m years

The value of the remaining endowment and insurance has already been found to be $A_{x+m\,\overline{n-m}|}$; the value of the premiums is $P_{x\,\overline{n}|}\,(1 + |_{n-m-1}a_{x+m})$. The equation becomes

$$A_{x+m\,\overline{n-m}|} = {}_{m}V\,P_{x\,\overline{n}|} + P_{x\,\overline{n}|}\,(1 + |_{n-m-1}\,a_{x+m})$$
$${}_{m}V\,P_{x\,\overline{n}|} = A_{x+m\,\overline{n-m}|} - P_{x\,\overline{n}|}\,(1 + |_{n-m-1}\,a_{x+m})$$

NET VALUATION, ANNUITIES, INSURANCES AND ENDOWMENTS. RETROSPECTIVE METHOD.

THE prospective method of valuation is the measure of what is required, together with future net premiums, to meet future costs of insurance, endowments or annuities.

But there must have been accumulated from past premiums an amount sufficient to meet this requirement.

If there has not been accumulated enough, it means that premiums were not adequate, in consequence of which, first, not enough has been accumulated from past premiums and, second, yet more is required to make good the deficiencies of future premiums.

If the premiums have been precisely correct, according to the assumed standards, it follows that, according to the same standards, just enough will be left of premiums already paid, with interest realized thereon, to equal the required reserve.

We may, therefore, ascertain the amount of the net reserve by considering only premiums paid. To ascertain the aggregate reserve in a company by this method, we would improve the premiums by interest at the rate employed in computing rates and would deduct at the proper intervals for costs of insurance, endowments or annuities, according to the tables. The result would be the aggregate reserve.

The principal value, however, of the retrospective system lies in its convenient application to the computation of individual reserves, as follows:

To the net premium of the first year of insurance add interest for one year and deduct the tabular insurance cost of the actual net insurance for one year at that age; the remainder is the terminal reserve. Add to this reserve the net premium for the second year and repeat the process. The "actual net insurance" is the difference between the face of the policy and the accumulation at the end of the year, the hypothesis being that death-claims are paid at the close of the insurance year.

A frequent and convenient though by no means necessary form of this computation is to deduct the tabular cost, discounted one year, at the beginning of the year and then improve only the remainder at interest. This is equivalent to the other form and offers no especial advantages; still it is frequently employed.

What makes the retrospective system of so peculiar value is that through it the values of all sorts of policies may be derived by one formula. We have already seen that a separate and distinct formula was required for each form of annuity, insurance or endowment. Other formulas would be required for every other distinctive policy and some of these formulas would be complex and unwieldy.

But the retrospective system disregards the form of the insurance almost entirely. Given two policies for the same amount with the same net premium, the reserve would be the same after the same number of years, call them what you will, the ages at entry being the same.

The simple arithmetical form of computing the reserves by this plan will answer as well as any; but it has been developed algebraically as follows:

Let ${}_1V_x$ be the value at end of one year, c_x be the discounted risk of one dollar at age x and P_x be the premium paid. Then,

$$1 : c_x :: 1 - {}_1V_x : c_x - c_x\,({}_1V_x)$$

The last factor in this proportion is thus the discounted value of the actual risk run by the company. Subtracting this value from the net premium P_x and accumulating the remainder at the assumed rate of interest, i, we have:

$$ {}_1V_x = (1+i)\,[P_x - c_x + c_x\,({}_1V_x)\,] $$
$$ {}_1V_x = (1+i)\,(P_x - c_x) + (1+i)\,c_x\,({}_1V_x) $$
$$ {}_1V_x\,[1 - (1+i)\,c_x\,] = (1+i)\,(P_x - c_x) $$
$$ {}_1V_x = \frac{1+i}{1-(1+i)\,c_x}\,(P_x - c_x) $$

We have $q_x = \frac{d_x}{l_x} = \frac{l_x - l_{x+1}}{l_x} = 1 - \frac{l_{x+1}}{l_x}$. And $p_x = \frac{l_{x+1}}{l_x}$, whence $q_x = 1 - p_x$. Analyzing the value of c_x, we find $c_x = v\frac{d_x}{l_x} = v\,q_x = \frac{q_x}{1+i}$ and $1 - (1+i)\,c_x = 1 - (1+i)\frac{q_x}{1+i} = 1 - q_x = p_x$. Substituting this value, we have:

$$ {}_1V_x = \frac{1+i}{p_x}\,(P_x - c_x) $$

Compute the value of $\frac{1+i}{p_x}$ and call it u_x and the equation becomes

$$ {}_1V_x = u_x\,(P_x - c_x) $$

Make a table of values of u_x for convenience, by the following formula:

$$ u_x = \frac{1+i}{p_x} = \frac{1+i}{\frac{l_{x+1}}{l_x}} = (1+i)\,\frac{l_x}{l_{x+1}}. $$

Multiplying numerator and denominator by v^{x+1}, which does not alter the value, we have

$$ u_x = \frac{(1+i)\,v^{x+1}\,l_x}{v^{x+1}\,l_{x+1}} = \frac{v^x\,l_x}{v^{x+1}\,l_{x+1}} = \frac{D_x}{D_{x+1}} $$

The value at the beginning of the second year is the value at the end of the first (${}_{1}V_x$) plus the second premium, P_x. We thus get by analogy:

$${}_{2}V_x = u_{x+1} ({}_{1}V_x + P_x - c_{x+1}) \text{ and, generally, }$$
$${}_{n+1}V_x = u_{x+n} ({}_{n}V_x + P_x - c_{x+n})$$

This formula will accurately give the values for any form of insurance involving a level premium and a level benefit. Its only inconvenience is that to give the value in n years the value in n — 1 years must be known. Practically this is usually no disadvantage at all, the course being commonly to compute all the values in a series and tabulate them for future reference.

The formula, however, requires nearly the same labor as the arithmetical formula first mentioned, almost its only superiority being that the actual insurance does not need to be computed for each year separately. It has come to be known as Wright's Accumulation Formula.

Elizur Wright, the inventor of the formula, also extended it to cover reserves at the end of any month in the policy year, thus:

To find the value for the nearest completed month, m, the formula becomes:

$${}_{n-\frac{12-m}{12}}V_x = {}_{n-1}V_x + P_x - \frac{m}{12} ({}_{n-1}V_x + P_x - {}_{n}V_x)$$

The point is that $({}_{n-1}V_x + P_x - {}_{n}V_x)$ represents the net waste or deduction from the accumulation during the year, which may be positive if the mortality charge exceeds the interest or negative if the interest exceeds the mortality charge; the theory is that $\frac{1}{12}$ of this waste must be charged off the sum of the value at the beginning of the year to get at the intermediate value. The same result may be arrived at by adding to the value at the end of the year the portions of the waste not yet made, thus:

$$ {}_{n-\frac{12-m}{12}}V_x = {}_nV_x + \frac{12-m}{12}\left({}_{n-1}V_x + P_x - {}_nV_x\right) $$

Another formula which proceeds on a somewhat similar principle to Wright's Accumulation Formula, is known as Fackler's Formula, from the name of its inventor, David Parks Fackler. It might with propriety be called a distribution formula, being based upon the idea of distributing or apportioning the aggregate reserve. Given l_{x+n} lives with individual reserves at the end of the last year of ${}_nV_x$ each, making an aggregate of $(l_{x+n})\ {}_nV_x$; and individual premiums of P_x or an aggregate of $l_{x+n}\ P_x$. By the close of the year these aggregate sums will be increased by i interest and will then be diminished by d_{x+n} death-claims, making the aggregate reserve at the end of the year stand as follows:

$$ l_{x+n+1}\left({}_{n+1}V_x\right) = \left[\left(l_{x+n}\right)\left({}_nV_x + P_x\right)\right](1+i) - d_{x+n} $$

$$ {}_{n+1}V_x = \frac{(1+i)\left(l_{x+n}\right)}{l_{x+n+1}}\left({}_nV_x + P_x\right) - \frac{d_{x+n}}{l_{x+n+1}} $$

$$ = \frac{1+i}{p_{x+n}}\left({}_nV_x + P_x\right) - \frac{d_{x+n}}{l_{x+n+1}} $$

$$ = u_{x+n}\left({}_nV_x + P_x\right) - \frac{d_{x+n}}{l_{x+n+1}} $$

Multiplying the numerator and denominator of $\frac{d_{x+n}}{l_{x+n+1}}$ by v^{x+n+1} which does not alter the value, we get $\frac{v^{x+n+1}}{v^{x+n+1}}\frac{d_{x+n}}{l_{x+n+1}}$ which is, in commutation terms, $\frac{C_{x+n}}{D_{x+n+1}}$. Expressing this value by k_{x+n}, which values may be tabulated, and substituting we get:

$$ {}_{n+1}V_x = u_{x+n}\left({}_nV_x + P_x\right) - k_{x+n} $$

NOTE.—The reserve usually employed in valuation is the "mean" or "midyear" reserve, it being assumed that insurances average a half year beyond their anniversary. An approximate formula is

$$ {}_{n-\frac{1}{2}}V_x = \frac{{}_{n-1}V_x + P_x + {}_nV_x}{2} $$

The following retrospective formula is due to Emory McClintock:

Starting from:

$$ l_{x+n+1}\,({}_{n+1}V_x) = [\,(l_{x+n})\,({}_nV_x + P_x)]\,(1+i) - d_{x+n} $$

$$ {}_{n+1}V_x = ({}_nV_x + P_x)\,u_{x+n} - \frac{d_{x+n}}{l_{x+n+1}} $$

$$ = ({}_nV_x + P_x)\,u_{x+n} - \frac{l_{x+n} - l_{x+n+1}}{l_{x+n+1}} $$

$$ = ({}_nV_x + P_x)\,u_{x+n} - \frac{l_{x+n}}{l_{x+n+1}} + 1 $$

$$ = ({}_nV_x + P_x)\,u_{x+n} - v\,u_{x+n} + 1 $$

$$ = 1 - [\,v - ({}_nV_x + P_x)]\,u_{x+n}. $$

This is susceptible of a special explanation. The actual net risk at the beginning of the n + 1th year = v − (${}_nV_x$ + P_x). Progressed to the end of the year, this becomes $[v-({}_nV_x + P_x)]\,u_{x+n}$. The n+1th reserve = 1 — actual net risk = $1-[v-({}_nV_x + P_x)]\,u_{x+n}$.

NON-CONTINUOUS RETROSPECTIVE VALUATION.

The formulas heretofore given enable the value of ${}_{n+1}V_x$ to be computed, only in case ${}_nV_x$ has already been computed. In other words they are suitable only for computing a continuous series of values.

A formula may, however, be developed for computing values by a retrospective process which will give ${}_{n+1}V_x$ without first obtaining ${}_nV_x$.

We have

$$ {}_n|a_x = \frac{N_{x+n}}{D_x} $$

in which make n negative and we get:

$$ {}_{-n}|a_x = \frac{N_{x-n}}{D_x} = \frac{(N_{x-n} - N_x) + N_x}{D_x} $$

The meaning of this is that the value is that of an annuity taken at age x—n and on which all past payments have been permitted to accumulate—i. e. ${}_{-n}|a_x =$ this accumulated value plus the present value of an annuity at age x.

We have

$$ E_x = \frac{D_{x+1}}{D_x}. $$

$$ \therefore E_x^{-1} = \frac{D_x}{D_{x+1}} = u_x $$

And

$$ {}_2E_x = \frac{D_{x+2}}{D_x} $$

$$ {}_2E_x^{-1} = \frac{D_x}{D_{x+2}} $$

After n years, the values of annuity payments not drawn will accumulate as follows:

The first payment $= {}_nE_x^{-1} = \frac{D_x}{D_{x+n}}$

The second payment $= {}_{n-1}E_{x+1}^{-1} = \frac{D_{x+1}}{D_{x+n}}$

And so on.

The entire accumulation is called a **foreborne temporary annuity due** (or, in America, foreborne immediate temporary annuity) and, as no symbol is assigned to it in Great Britain, we may extend the American symbol, u_x into ${}_nu_x$. We have, then:

$$\begin{aligned} {}_nu_x &= {}_nE_x^{-1} + {}_{n-1}E_{x+1}^{-1} + \ldots . \; E_{x+n-1}^{-1} \\ &= \frac{D_x + D_{x+1} + \ldots .. \; D_{x+n-1}}{D_{x+n}} \\ &= \frac{N_{x-1} - N_{x+n-1}}{D_{x+n}} \end{aligned}$$

Since ${}_nu_x =$ an accumulation of a foreborne temporary annuity due of 1 each year for n years, ${}_nu_x \, P_x =$ the accumulation of the net annual premiums for the same period.

In like manner, let us suppose that the death claims have been accumulated at interest for n years, we shall have for each policy's share the following:

At end of 1 year, $\frac{d_x}{l_{x+1}} = k_x = \frac{C_x}{D_{x+1}}$

At end of 2 years, $\frac{C_x}{D_{x+2}} + \frac{C_{x+1}}{D_{x+2}}$

And so on.

Since no symbol is assigned to such an accumulation in the International notation, we may extend the American symbol k_x into ${}_nk_x$. We have, then:

$$\begin{aligned} {}_nk_x &= \frac{C_x + C_{x+1} + \ldots . \; C_{x+n}}{D_{x+n}} \\ &= \frac{M_x - M_{x+n}}{D_{x+n}} \end{aligned}$$

But the reserve under a policy after n years equals the difference between the accumulated value of the premiums and the accumulated value of the policy's share of death-claims; that is:

$$_{n}V_x = P_x\,({}_{n}u_x) - {}_{n}k_x.$$

$$_{n+1}V_x = P_x\,({}_{n+1}u_x) - {}_{n+1}k_x$$

A policy's share of the accumulation of death claims under an insurance increasing by 1 each year may also be represented in a similar expression, viz.

At end of 1 year $= \dfrac{C_x}{D_{x+1}}$

At end of 2 years $= \dfrac{C_x}{D_{x+2}} + \dfrac{2\,C_{x+1}}{D_{x+2}}$

And so on.

Since also no symbol is assigned to such an accumulation in Great Britain, we may employ the symbol ${}_{n}I_x$. We have then:

$$_{n}I_x = \frac{C_x + 2\,C_{x+1} + 3\,C_{x+2} + \ldots\, n\,C_{x+n}}{D_{x+n}}$$

$$= \frac{R_x - R_{x+n} - n\,M_{x+n}}{D_{x+n}}$$

The formula for the value after n years of an increasing insurance of 1, with annual premiums, becomes:

$$_{n}V = \pi_x\,({}_{n}u_x) - {}_{n}I_x).$$

And the formula for a return premium insurance with net extra premium ϕ_x, becomes:

$$_{n}V = (P+\varphi)\,{}_{n}u_x - ({}_{n}k_x + P'\,{}_{n}I_x).$$

This formula is by all means the most convenient one for valuing return premium policies when values of ${}_{n}u_x$, ${}_{n}k_x$ and ${}_{n}I_x$ have been tabulated.

VALUATION—ADDITIONAL FORMULAS.

SUPPOSE the reserve $_{n+1}V_x$ to be known, to derive $_nV_x$. It is clear that at the beginning of the (n + 1)th year, the premium having been paid, there was in hand $_nV_x + P_x$, which sum was just sufficient with its interest to cover the insurance cost and produce $_{n+1}V_x$, the reserve at the end of the year. It follows that it is the equivalent of the present values of the insurance cost and the terminal reserve, thus:

$$_nV_x + P_x = v\,[q_{x+n} + (p_{x+n})\,(_{n+1}V_x)]$$

$$_nV_x = v\,[q_{x+n} + (p_{x+n})\,(_{n+1}V_x)] - P_x$$

This formula is general, applying to all annual premium policies.

To get the reserve $_nV_x$ in terms of the premium P_x and the annuity, a_{x+n}. If the insurance were payable at once, its value would be 1; not being payable until death of insured now aged x + n, we must deduct v i $(1 + a_{x+n})$, the value of the interest that 1 would earn before death occurs and also the value of premiums still to be paid, $P_x\,(1 + a_{x+n})$, thus:

$$_nV_x = 1 - (P_x + d)\,(1 + a_{x+n})$$

Given the premium, P_x, the premium, P_{x+n}, and the single premium, A_{x+n}, to get a formula confined to these terms. A person aged x + n, applying for insurance, would pay a premium P_{x+n} for an insurance of 1; a premium of P_x would purchase an insurance of $\frac{P_x}{P_{x+n}}$. But the insurance, issued n years ago, is for 1; it follows that the reserve held to secure it must be sufficient to pay for an insurance of $1 - \frac{P_x}{P_{x+n}}$, thus:

$${}_nV_x = A_{x+n}\left(1 - \frac{P_x}{P_{x+n}}\right)$$

To get a formula in terms of a_x and a_{x+n} Suppose a person aged x invests 1 in an immediate life annuity; it will buy $\frac{1}{1 + a_x}$ a year in advance. Of this amount v i is equivalent to the value of the interest and the remainder is equivalent to the 1, which disappears or reverts to the insurance company on the death of x. Therefore, it is also equal to P_x. The whole annuity, then, is v i + P_x ; and if P_x is paid from it each year for an insurance of 1, the joint values of the insurance and annuity will always be 1, since they secure, first, the equivalent of the interest on 1 until death, and upon death, 1. After n years, then, the value of the insurance is 1, less the remaining annuity, thus:

$${}_nV_x = 1 - \left(\frac{1}{1 + a_x}\right)(1 + a_{x+n}) = 1 - \frac{1 + a_{x+n}}{1 + a_x}$$

$$= \frac{a_x - a_{x+n}}{1 + a_x}$$

To get a formula in terms of A_x and A_{x+n}. Substituting in the foregoing equation the values of a_{x+n} and a_x, as per the formula:

$$a_x = \frac{1 - A_x}{v\,i} - 1, \text{ we get:}$$

$${}_nV_x = 1 - \frac{1 + \frac{1 - A_{x+n}}{v\,i} - 1}{1 + \frac{1 - A_x}{v\,i} - 1} = 1 - \frac{1 - A_{x+n}}{1 - A_x}$$

$$= \frac{A_{x+n} - A_x}{1 - A_x}$$

To get a formula in terms of P_x and P_{x+n}. Substituting in the same equation the values of a_{x+n} and a_x, as per the formula:

$$a_x = \frac{1}{P_x + d} - 1, \text{ we get:}$$

$$ {}_{n}V_{x} = 1 - \frac{1 + \dfrac{1}{P_{x+n} + vi} - 1}{1 + \dfrac{1}{P_{x} + vi} - 1} = 1 - \frac{P_{x} + vi}{P_{x+n} + vi} $$

$$ = \frac{P_{x+n} - P_{x}}{P_{x+n} + vi} $$

To find the values of net reserve ${}_{n}V_{x}$ and of gross reserve ${}_{n}V'_{x}$ in terms of the other. One of our simplest formulas for the net reserve is:

$$ {}_{n}V_{x} = (P_{x+n} - P_{x})(1 + a_{x+n}) $$

The interpretation, as hitherto explained, is that to make good the difference between the net premium at age attained and the net premium at age at entry, we must have the value of an immediate annuity equal to this difference. In this formula we may make a complete separation, thus: $P_{x+n}(1 + a_{x+n})$ = the value of the future insurance and $P_{x}(1 + a_{x+n})$ = the value of the future premiums. By gross valuation, the value of future insurance is not altered, but the value of future premiums is made $P'_{x}(1 + a_{x+n})$, the value of an immediate annuity equal to the office premium instead of the net premium. The formula thus becomes:

$$ {}_{n}V'_{x} = (P_{x+n} - P'_{x})(1 + a_{x+n}) $$

Let $P'_{x} = P_{x} + \Phi$ and substituting, we get:

$$ {}_{n}V'_{x} = (P_{x+n} - P_{x} - \Phi)(1 + a_{x+n}) $$
$$ = {}_{n}V_{x} - \Phi(1 + a_{x+n}) $$

In words, the gross reserve equals the net reserve, less an immediate annuity of the difference between the net and gross premiums. This is on the basis that the gross premium exceeds the net; otherwise the sign of Φ is changed to +.

Conversely:

$$ {}_{n}V_{x} = {}_{n}V'_{x} + \Phi(1 + a_{x+n}) $$

TRUE RETROSPECTIVE RESERVES.

In dealing with premiums and loading, it was pointed out that the customary system of computing the expense provisions is crude and inaccurate and, especially, non-actuarial and unscientific, in that it makes no adequate provision for the initial expense. It was suggested that this could be remedied by increasing the computed net premium by an annuity equal to the sum required to cover this expense and then deducting the sum required from the first of these net premiums, leaving the remainder only as a real net premium for the first year.

In premium-computations this might not seem important since it is easy to add this annuity in making up the gross or office premiums; and then, since the premiums are adequate, we need not bother about their division into net premiums and loading.

But when it comes to computing reserves the case is different. These are computed on the basis of the net premiums, the loading not merely being accounted adequate to cover the expenses, but also able to cover them when they are incurred, *i. e.*, not only equivalent to them, but actually discharging them. But this is just what it will not do when the customary system is employed.

The formulas for reserves are based on this idea, viz., that the net premium is progressed at interest and the cost of insurance deducted, the remainder being the reserve.

This gives a reserve at the end of the first and of every other year but the last, larger than the actual premiums, after providing for the foreseen expenses and losses, will accumulate on the basis of the interest and mortality assumptions on which both premiums and reserves are computed. If companies had no salvage on mortality or expense estimates and no excess interest, they could not qualify under such requirements; and a new company with a large business cannot do so, anyhow. Since the reserve is intended to accurately measure the liability on the supposition that there is no gain or salvage, it follows that it is not a fair test.

Thomas Bond Sprague, ex-president of the Faculty of Actuaries of Scotland, presented this matter at the Congress of Actuaries in 1895, and it was practically the unanimous view of the distinguished actuaries there present that the theoretical reserve system needs such modification as will make it square with the facts.

The true reserves, actual net premiums having been determined by the formulas providing for initial cost, are not more difficult to compute than other reserves. They may be readily calculated by the retrospective accumulation plan by simply dealing with the first year net premium (${}^1P = {}^2P - F$) and thereafter with the subsequent net premiums (${}^2P = P + f$). The same values of u_x will apply.

On the prospective plan the formulas require no alteration except the substitution of the new subsequent net premium (${}^2P = P + f$) for the old or flat net premium.

Or the true reserve may be derived from the theoretical reserves by the following simple process: deduct from the ascertained net theoretical reserve the value of the immediate annuity added to the net premium as an equivalent to the initial cost, thus:

$${}_nT\,V_x = {}_nV_x - f\,(1 + a_{x+n})$$

If the premiums be payable for a limited term only, the formula becomes

$${}_nT\,V_x = {}_nV_x - f\,(1 + |_{m-n-1}a_{x+n})$$

In another respect the theoretical reserves often differ from the true reserves, viz., in the case of all paid-up insurances and also of all insurances with limited premiums during the premium-paying period as well. This for the reason that an intelligently computed loading should provide for expenses all through the life of the policy and not merely while premiums are being paid. The net reserve system which takes into account only death losses does not make this provision.

Let ϕ be the annual provision for expenses after premium-paying ceases and ϕ' be the part of the loading during the premium-paying period intended to accumulate this provision. Then to find the true reserve after the premium-paying period, the theoretical reserve must be increased by the value of an annuity of ϕ. In the case of whole life insurance this becomes:

$${}_{n}T\,V_x = A_{x+n} + \phi\,(1 + a_{x+n}).$$

In the case of any insurance terminating in a less period, the formula is:

$${}_{n}T\,V_x = {}_{n}V_x + \phi\,(1 + |_{m-n-1}a_{x+n})$$

During the premium-paying period the theoretical reserve is increased by the amount of the accumulation of ϕ' up to the date of valuation, thus:

$${}_{n}T\,V_x = {}_{n}V_x + \phi'\,{}_{m}u_x.$$

MEAN AND OTHER AD INTERIM RESERVES.

All the foregoing has to do with "terminal reserves," i. e., reserves for integral years of insurance, so that n in all the formulas is an integral number of years. There remains to consider how these formulas are modified when the term is an integer plus a fraction—n + k years, in which n is integral and k fractional.

The prospective formula for a whole life reserve now becomes;

$${}_{n+k}V_x = A_{x+n+k} - P_x\,({}_{1-k}|a_{x+n+k})$$

When $k = \frac{1}{2}$, this becomes:

$${}_{n+\frac{1}{2}}V_x = A_{x+n+\frac{1}{2}} - P_x\,({}_{\frac{1}{2}}|a_{x+n+\frac{1}{2}})$$

$A_{x+n+\frac{1}{2}}$ may be taken as approximately equal to $\frac{A_{x+n}+A_{x+n+1}}{2}$ and $\frac{1}{2}|a_{x+n+\frac{1}{2}}$ as $\frac{1}{2} + \frac{a_{x+n}+a_{x+n+1}}{2}$.

The prospective formula for the reserve of a whole life insurance, paid-up by t premiums, becomes when n < t:

$${}_{n+k}V_t\,P_x = A_{x+n+k} - {}_tP_x\,({}_{1-k}|a_{x+n+k\,\overline{t-(n+1)}|})$$

When $k = \frac{1}{2}$, this becomes :

$${}_{n+\frac{1}{2}}V_t\,P_x = A_{x+n+\frac{1}{2}} - {}_tP_x\,({}_{\frac{1}{2}}|a_{x+n+\frac{1}{2}\,\overline{t-(n+1)}|})$$

As before $A_{x+n+\frac{1}{2}}$ may be taken as approximately equal to $\frac{A_{x+n}+A_{x+n+1}}{2}$ and $\frac{1}{2}|a_{x+n+\frac{1}{2}\,\overline{t-(n+1)}|}$ may be taken as $\frac{1}{2} + \frac{a_{x+n\,\overline{t-(n+1)}|} + a_{x+n+1\,\overline{t-(n+2)}|}}{2}$,

The prospective formula for the reserve of an endowment insurance in m years, becomes:

$${}_{n+k}V_{x\,\overline{m}|} = A_{x+n+k\,\overline{m-(n+k)}|} - P_{x\,\overline{m}|}\,({}_{1-k}|a_{x+n+k\,\overline{m-(n+1)}|})$$

When $k = \frac{1}{2}$, this becomes:

$${}_{n+\frac{1}{2}}V_{x\,\overline{m}|} = A_{x+n+\frac{1}{2}\,\overline{m-(n+\frac{1}{2})}|} - P_{x\,\overline{m}|}\,({}_{\frac{1}{2}}|a_{x+n+\frac{1}{2}\,\overline{m-(n+1)}|})$$

In this $A_{x+n+\frac{1}{2}\,\overline{m-(n+\frac{1}{2})}|}$) may be taken as equal to

$\frac{A_{x+n\,\overline{m-n}|} + A_{x+n+1\,\overline{m-(n+1)}|}}{2}$ and ${}_{½|}a_{x+n+½\,\overline{m-(n+1)}|}$ as $½ + \frac{a_{x+n\,\overline{t-(n+1)}|} + a_{x+n+1\,\overline{t-(n+2)}|}}{2}$.

When k is a fraction differing from ½, A_{x+n+k} is taken as $A_{x+n} + k\,(A_{x+n+1} - A_{x+n})$ and ${}_{1-k|}a_{x+n+k}$ as $k + k a_{x+n+1} + (1-k)\,a_{x+n}$. In such case ${}_{1-k|}a_{x+n+k\,\overline{t-(n+1)}|}$ is taken as $k + k\,a_{x+n\,\overline{t-(n+1)}|}) + (1-k)\,(a_{x+n+1\,\overline{t-(n+2)}|})$.

Reserves when $k = ½$ are known as "mean reserves" and are especially important in the United States and Canada, where state department valuations are usually made on the basis of mean reserves for all policies, it being assumed that the policies of each calendar year have been issued on the average on June 30.

The terminal reserves having been computed however, the reserves for a fractional part of the year are obtained very easily by the formula:

$${}_{n+k}V = {}_{n}V + P + k\,[{}_{n+1}V - ({}_{n}V + P)]$$

in which $k\,[{}_{n+1}V - ({}_{n}V + P)]$ may be either positive or negative.

Mean reserves are obtained yet more easily by the approximate formula:

$${}_{n+½}V = \frac{{}_{n}V + {}_{n+1}V + P}{2}.$$

This is the method generally adopted in practice.

RESERVES. SELECT. SELECT AND ULTIMATE.

All of the foregoing formulas, suitably modified, are applicable in computing reserves by select tables, the net annual premium, the net single premium and the annuity all being according to the select tables. Usually such reserves will be found to be larger than the reserves by an aggregate table representing the same experience and also than the reserves by an ultimate table drawn from the same experience.

In Great Britain, net premiums by the select tables are frequently used in fixing the actual office premiums; but they are very rarely employed for computing reserves. The net premium for valuation is usually taken from an aggregate table; and valuation is then made, either by net single premiums and annuities from the same aggregate table throughout, or during the first five years by such and thereafter by net single premiums and annuities from the corresponding ultimate table.

The last mentioned method, using H^M and $H^{M(5)}$ as symbols for the tables, is known as H^M and $H^{M(5)}$ valuation and is by the following formulas for whole life reserves:

First five years, H^M throughout:

$${}_{n+k}V_x = A_{x+n+k} - P_x\,({}_{1-k}|a_{x+n+k})$$

Thereafter

$${}_{n+k}V_x = A^{HM(5)}_{x+n+k} - P^{HM}_x\,({}_{1-k}|a^{HM(5)}_{x+n+k}).$$

In the United States, the Actuaries and American Experience tables have been continued as standards by which to compute net premiums, because believed to be safe and to represent conservatively the mortality to be expected, after the benefits of selection wear off—in other words, to be tantamount to ultimate tables. These tables are aggregate tables, representing in the former case the experience of 17 British offices compiled about 60 years ago, and in the latter the early experience of the Mutual Life Ins. Co. of New York. The improved hygienic and sanitary conditions and closer selection by medical examination have so improved the mortality upon insured lives that an aggregate table representing recent experience would be much lower, and these tables are, perhaps, even somewhat higher than an ultimate table, drawn from recent experience, would be.

A mode of valuation which would represent the exact facts, the net premium being computed by an ultimate table, is found in a select and ultimate formula as follows:

Take the net annual premium by the ultimate table and the net single premium and annuity by the select table, which accurately measures the mortality to be expected in the early years of the policy, giving:

$$
\begin{aligned}
{}_{n}V_x &= A_{[x]+n} - P_x\, a_{[x]+n} \\
{}_{n+k}V_x &= A_{[x]+n+k} - P_x\, ({}_{1-k}|\, a_{[x]+n+k}) \\
{}_{n+\frac{1}{2}}V_x &= A_{[x]+n+\frac{1}{2}} - P_x\, (\tfrac{1}{2}\,|\, a_{[x]+n+\frac{1}{2}}) \\
&= \frac{A_{[x]+n} + A_{[x]+n+1}}{2} - P_x \left[\tfrac{1}{2} + \frac{a_{[x]+n} + a_{[x]+n+1}}{2}\right]
\end{aligned}
$$

After the period affected by selection elapses, the reserves by this method are identical with the reserves by the ultimate table; during the period affected by selection, they are lower and gradually approach the reserves by the ultimate table. The effect of the system is to allow the salvage on mortality as compared with the ultimate table, not as is done in Great Britain when select net premiums are used, in a reduction of the net premiums below those derived from the ultimate table, but as an offset to the large expenses the first year. The amount of the offset, as computed by existing select tables, is not far from what is conservatively required according to the usual company experience in the United States and in other countries. Since the salvage in mortality is due to new admissions and to medical selection, it seems eminently proper that it be first offset against the especial expenses incurred to secure the new insurances and to make the medical selections.

The "select and ultimate" method provides a safe net premium by a table representing conservatively the mortality to be expected when the benefits of fresh selection have worn off and it values by net single premiums and annuities which represent, as accurately as possible, the exact facts.

PRELIMINARY TERM RESERVES.

In 1863, Dr. Zillmer, a German actuary, suggested that if reserves on ordinary life policies (many more were then upon that plan than upon all others) were computed on the basis that the premium for the first year is for one year term insurance and that the ordinary life premium begins one year later at an age one year higher, such method would give sufficient reserves and allow a proper provision for initial expenses.

Many years later, Dr. Thomas Bond Sprague, the famous British actuary, took the same position, especially as to ordinary life policies; and within a few years, the same position has again been publicly taken by several eminent American actuaries.

On the continent of Europe, where the suggestion of Dr. Zillmer had most influence, the idea has been made use of by many companies, most frequently by modifications of the method of loading and by valuing by net premiums thus determined, after the manner described in the chapter, "Valuation: True Reserves." Very little use of the idea has been made in Great Britain, where valuations usually take place once in five years, but somewhat more in the colonies. Usually in these countries, where a compulsory system of *net* valuation has not been introduced, policies which are thus valued are not written in such manner as to be preliminary term contracts by their language.

In the United States within the last ten years, many companies have sought to avail themselves of Dr. Zillmer's suggestion by writing their policies as term insurance the first year. The policies have accordingly been generally so valued. The values are mean reserves; and, for term insurance for one year followed by whole life, are by the following formulas:

$$1\text{st year} = \frac{P^{1}_{x\,\overline{1}|}}{2}$$

$$n\text{th year} = \frac{{}_{n-2}V_{x+1} + {}_{n-1}V_{x+1} + P_{x+1}}{2}.$$

The·same formulas, unchanged for the first year and with suitable substitutions for subsequent years, may be used for preliminary term, followed by life insurance premiums limited to 9, 14 or 19 years, or by endowment insurance maturing at death or in 9, 14 or 19 years, for instance.

With the exception of certain companies operating under a special statute in Massachusetts, no companies in the United States have policies valued as preliminary term which do not contain provisions calling for such valuations.

By each of the actuaries, already mentioned, who suggested this form of valuation, objection was made to its application to limited-payment life and endowment insurance policies, without modification. On the continent, this was generally avoided by the adoption of some adjustment of the loading, as set forth in the chapter on "Valuation: True Reserves." In some of the colonies of Great Britain and in the United States, suitable modifications have in some cases been made, by making the premium for the preliminary term insurance exactly equal to the corresponding whole life premium, at the age next higher, and by putting up a pure endowment reserve from the excess of the first net annual premium over this preliminary term premium.

The process, in the case of an endowment in m years, is as follows:

Let $\pi = \frac{1 - {}_{m-1}V_{x+1}}{{}_{m}u_x}$

The usual net premium P_{x+1} will provide the insurance during m years, and also the sum of ${}_{m-1}V_{x+1}$ at the end of that period. The sum to be provided from the pure endowment is therefore $(1 - {}_{m-1}V_{x+1})$; and the annual premium which will accumulate to this quantity is $\frac{1 - {}_{m-1}V_{x+1}}{{}_{m}u_x}$ as already given.

The first year's net premium, ${}^{1}P = P^{1}_{x:\overline{1}|} + \pi$.

Second and subsequent net premiums ${}^{2}P = P_{x+1} + \pi$.

1st terminal reserve $= \pi\, u_x$.

nth terminal reserve $= {}_{n-1}V_{x+1} + \pi\, {}_{n}u_x$.

1st mean reserve $= \frac{P^{1}_{x:\overline{1}|} + \pi + \pi\, u_x}{2}$

nth mean reserve $= \frac{{}_{n-2}V_{x+1} + {}_{n-1}V_{x+1} + P_{x+1} + \pi({}_{n}u_x + {}_{n-1}u_x) + \pi}{2}$

In the case of a limited-premium whole life insurance, by t premiums, the formula for π becomes:

$$\pi = \frac{A_{x+t} - {}_{t-1}V_{x+1}}{{}_{t}u_x}$$

The other formulas remain unchanged.

This modified form of preliminary term valuation when made by an "ultimate" table gives an allowance to offset first year's expenses, not much in excess of the allowance by a "Select and Ultimate" valuation; but the reserves do not become the full ultimate reserves (not preliminary term) after the selection period has elapsed.

ACTUAL INSURANCE.

If a life is insured for one year, for instance,—a term which is usually taken as the unit,—the premium paying for insurance for that term and no longer, it is expected that his net premium, together with others, will pay the losses of the year. This premium he has lost, whether he survives or dies, it being exacted of all alike; it will be entirely absorbed if the company's mortality experience corresponds to the tables.

In betting parlance, the company is both a bettor and a stakeholder. If the insured wins, the company returns the stake, the premium, and pays the bet, the amount insured less the premium, making a total return of the amount insured.

Companies in rate-computation treat the loading as immediately available for expenses and disregard it in all other computations, except ascertainment of surplus. Strictly speaking, having the nature of the bet clearly in view, the company ought to treat the whole premium as "not earned" for the year when death occurs and charge others for the cost only of the insurance in excess of the premium with accrued interest to the end of the year, the actual insurance being: $1 - P'_x (1 + i')$.

The practice, however, is to treat the loading as an earned portion. Even according to this view it would appear that in computations all the net premium should be treated as a bet returned and only the remainder charged against other policies, the actual insurance becoming $1 - P_x (1 + i')$.

But a moment's reflection shows that if all the insured be charged out of their premiums their *pro rata* shares of the total death-losses there will be no unearned portion to return, the whole net premium being thus charged out; and the cost to the others will be precisely the same as if the other process were followed, since all the net premiums on policies terminated by death are applied to pay death-losses in either case.

Consequently as it conduces to ease of computation, actuaries consider the actual insurance under one year term policies to be the face thereof.

But suppose an insurance with a premium such that if the insured survives, only part is "earned" and the remainder is to pay for insurance through all the remaining years of life; then the actual insurance is face of the policy, less the net single premium one year later, thus: $1 - A_{x+1}$.

The case of an annual or limited premium policy differs from this only in this, that the unearned portion is now the reserve, whatever that may be, and the actual insurance becomes $1 - {}_nV_x$, which may be taken as a general formula for the tabular actual insurance, ${}_nV_x$ becoming 0 when the term is one year and A_{x+n} when the insurance is for life at a single-premium.

The foregoing formula rests upon the view that the accumulation, if any, in excess of the reserve, is paid over in cash, additions to the insurance or reverts to the survivors, and so need not be considered as an abatement of the net amount to be paid by the company.

Let us suppose the fund to exceed the reserve, the accumulation then would be as follows: to the fund at the beginning of the year add the premium, deduct the expense, increase the remainder by interest and then deduct the cost of the insurance; the remainder is the fund at the close of the year.

It was possible when dealing with reserves only to compute in advance the reserves without direct reference to the cost of insurance; consequently the alternative of charging out the cost instead of applying the whole net premium in reduction of the loss could be followed.

This is not really feasible in calculations dealing with an actual fund, actual interest and actual mortality except by circumlocutions which are wholly unnecessary. Consequently the course should be to apply the whole fund to diminish the loss, compute the cost of insurance only for the net loss and only against policies not terminated by death. This, as we have already seen, is equivalent to the other plan, the same sums in either case being applicable to pay losses. Under this plan the real, actual insurance becomes $1 - ({}_{n-1}F_x + P'_x - \text{Exp.})\ (1 + i')$, F being the symbol for the fund and i′ for rate of actual interest; or, in these days of immediate settlements, $1 - ({}_{n-1}F_x + P'_x - \text{Exp.}).\ (1 + i)^{1/2}$

This last formula will be found especially useful in treating the cases of all insurances, such as in assessment companies, where the premium, while in excess of current requirements, is not computed according to some standard so as to furnish adequate reserves. By means of this formula the actual insurance may be charged for and the actual accumulation may be properly calculated. Because of the absence of such a formula many associations treat the amount insured as the actual insurance, without regard to the accumulation, and thus unnecessarily increase their own embarrassment as the persons insured grow older.

COST OF INSURANCE.

THAT portion of the net premium which is " earned " at the end of any year if the insured survives is called the " cost of insurance." It is the part of the money paid by the insured, which, with the expense payment, constitutes his bet against the " actual insurance " that he will not die that year.

Being a bet against the "actual insurance," it follows that it is a function of that sum, *i. e.*, that it is figured upon it.

In the case of a one-year insurance, if the insured survives the year, the company has earned the entire net premium $\frac{v\, d_x}{l_x}$ with interest thereon, or, interest being at the assumed rate, $\frac{v\, d_x}{l_x}\,(1 + i) = \frac{d_x}{l_x}$.

A series or table of the values of $\frac{d_x}{l_x}$, then, giving x successively the value of each of the ages, shows the costs of an actual insurance of 1 at each age. Such values are tabulated under the symbol q_x.

Costs of insurance which correspond to this tabular experience are known as "tabular costs of insurance," and are useful in computing reserves, etc. They are designated by the symbol ${}_nK$, followed by a symbol signifying the sort of policy as ${}_nK\,A_x$, the cost of insurance in the nth year of a single premium insurance issued at age x, etc.

These values are based upon the tabular actual insurance, $1 - {}_nV_x$. If $\frac{d_{x+n}}{l_{x+n}}$ or q_{x+n} is the cost of an insurance of 1 at age x + n, it follows that ${}_nK_x = (1 - {}_nV_x)\,\frac{d_{x+n-1}}{l_{x+n-1}}$

which formula is absolutely general for insurances for a level amount. It may be made still more general so as to cover increasing or decreasing insurances by adopting the expression $(w\,d)_x$ for the whole benefit at death, thus:

$${}_nK_x = [\,(w\,d)_x - {}_nV_x\,]\,\frac{d_{x+n-1}}{l_{x+n-1}}$$

The costs of insurance for common policies according to most standards have been tabulated. Tables by the usual American standards are published in " Principles and Practice of Life Insurance."

The " actual cost of insurance," designated by the symbol ${}_nK'$, bears that ratio to the tabular cost which the actual losses, less reserves, do to the aggregate costs of insurance, thus:

$${}_nK' : {}_nK : : \text{losses} - \text{reserves} : \text{aggregate } K.$$

This is the usual way of computing it. It is sometimes varied by comparing actual to expected losses, which will usually give approximately the same results, thus:

$${}_nK' : {}_nK : : \text{actual losses} : \text{expected losses}.$$

A company which is big enough and has a broad enough experience to form a mortality table of its own might use values of q_x drawn therefrom to make its K's.

The formula for the actual cost of an insurance for one year is

$$K'_x = \frac{d'_x}{l'_x} = q'_x$$

By this is not meant the actual deaths at the age x, divided by the actual entrants; but merely values drawn from the following proportion:

$$\frac{d'_x}{l'_x} : \frac{d_x}{l_x} : : \text{losses} - \text{reserves} : \text{aggregate } K.$$

EXPOSED TO RISK AND EXPECTED LOSSES.

If all insurances were issued on January 1st of each year and the terminations all took place on December 31st, it would be an easy matter to compute the "exposed to risk," the gross "expected losses" and the net "expected losses." For in order to get the "exposed to risk" it would be necessary only to add together the lives at age x on January 1st; or, if amounts are desired, the amounts of insurance in force at age x on January 1st. And the gross expected loss for that age would be this aggregate amount of insurance multiplied into q_x, and the net expected loss this aggregate amount of insurance less the aggregate reserves at the end of the year, into q_x.

But, in practice, the insurances are issued and terminate at various times in the year, and it is a matter of some difficulty to ascertain the "exposed to risk."

Let us assume that the insurances were issued on the average one-half year before December 31st and at exact ages, on the average, the age at issue being determined by nearest birthday. Also that withdrawals take place uniformly throughout the calendar year. Then on January 1st, there will be a group of lives, averaging $x+\frac{1}{2}$ years of age, formed of lives insured at age x a half-year ago, at age $x-1$ a year and a half ago, etc. Assign this the symbol $s_{x+\frac{1}{2}}$, meaning the survivors, entering upon this year in the xth year of their age as determined by policy years. Let n_x be the lives admitted at age x during the year and $w_{x+\frac{1}{2}}$, the lives withdrawing this year during the year x to $x+1$ of their age. Then

$$E_{x+\frac{1}{2}} = s_{x+\frac{1}{2}} + \frac{1}{2}\,(n_x + n_{x+1}) - \frac{1}{2}\,(w_{x+\frac{1}{2}} + w_{x+\frac{3}{2}})$$

But the actual withdrawals out of the survivors and entrants can be employed, instead of $\frac{1}{2}\,(w_{x+\frac{1}{2}} + w_{x+\frac{3}{2}})$

In the foregoing expression, x is intended to signify the average exact age in each case. A usual form for such a formula is:

$$E_{x+\frac{1}{2}} = s_x + \tfrac{1}{2}(n_x + n_{x+1}) - \tfrac{1}{2}(w_x + w_{x+1})$$

in which x means average exact age in $E_{x+\frac{1}{2}}$ and in n_x and n_{x+1}, only, and in all other expressions means in the x+1th year of age, as determined by policy years.

The deaths are not deducted, because in order to ascertain the "exposed to risk" of death, it is assumed that all deaths take place at the end of the year—for age x+½, at attained age x+$\frac{3}{2}$—and so that all lives, expiring by death, are exposed for a full year.

To ascertain the "expected deaths," $E_{x+\frac{1}{2}}$ must be multiplied by $q_{x+\frac{1}{2}}$.

If amounts, instead of lives, are entered in the computations, multiplying by $q_{x+\frac{1}{2}}$ will give the "gross expected loss." The "expected deaths" for each age x+½ or the "gross expected losses" for the same, when added together, may be compared with the actual deaths or the gross actual loss.

If the "net expected loss" is wanted, we must take into account that the reserves which fall in at death, will also need to be dealt with. We cannot take the mean reserves at age x+$\frac{3}{2}$ as approximately the reserves which fall in, as might at first be thought proper; for the reserves to fall in are only the terminal reserves at exact age x+1 for policies terminated by death on the average during the first half of the year and at exact age x+2 for policies terminated by death, on the average during the second half of the year. Therefore, we may take the mean between the terminal reserves for exact ages x+1 and x+2.

The "exposures" of these reserves may be computed by the same formula, $s_{x+\frac{1}{2}}$ being taken as the aggregate of the mean of the terminal reserves at average exact ages, $x+1$ and $x+2$, on the policies embraced in $s_{x+\frac{1}{2}}$ in finding the "gross expected loss;" $\frac{1}{2}\,n_x$ being one-half the first terminal reserves on policies issued during the year at age x; $\frac{1}{2}\,n_{x+1}$, the same on policies issued at age $x+1$; $\frac{1}{2}\,(w_{x+\frac{1}{2}} + w_{x+\frac{3}{2}})$ can, of course, be accurately computed, since they are the reserves actually falling in by withdrawal under policies included in $s_{x+\frac{1}{2}}$ in computing the gross expected loss.

When the "exposed reserves" for average exact age $x+\frac{1}{2}$ are computed, multiply by $q_{x+\frac{1}{2}}$ to find the "expected gain" by reserves falling in by death; sum for all values of $x+\frac{1}{2}$ and deduct from the "gross expected loss" to get the "net expected loss." Or the reserve exposure may be deducted from the risk exposure, before multiplying by $q_{x+\frac{1}{2}}$.

A rough approximation of the expected deaths or the gross expected loss during any year may be made by treating as the exposures at average exact age $x+\frac{1}{2}$, $\frac{1}{2}\,(s_{x+\frac{1}{2}} + e_{x+\frac{3}{2}})$—in which $e_{x+\frac{3}{2}}$ is the number (or amount insured) existing at the close of the year (corresponding to $s_{x+\frac{1}{2}}$ at the beginning) at the average exact age $x+\frac{3}{2}$.

A common mistake is to take as the exposures $\frac{1}{2}\,(s_{x+\frac{1}{2}} + e_{x+\frac{1}{2}})$ and multiply by q_x. It gives only a rough approximation at best and may be far out.

having been ascertained, an individual surplus account may be made up as follows:

Salvage on mortality.....	——% of "cost of insurance"	$—— .—	$—— .—
Salvage on loading.......	——% of "loading".........	—— .—	—— .—
Excess interest on reserve	——% of "mean reserve"...	—— .—	—— –
Total..			$—— .—

To this may be added gains from forfeitures and surrenders, which, while not covered by the "contribution formula," it would seem reasonable to divide in the ratio of the amount belonging to each policy that is subject to the risk of loss by forfeiture or upon surrender, thus:

Gains from discontinuances ——% of "mean amount risked" $——.— $——.—

The process may be restated, as follows: Increase the reserve from the previous year by the gross premium, less its share of the actual expenses; increase this sum by the rate of actual interest; diminish it by the actual "cost of insurance," *i. e.*, the tabular "cost of insurance," less the salvage thereon; this gives the entire fund at the close of the year; deduct the terminal net reserve and the remainder is surplus. Mr. Homans embodies this in the following formula, adopting B, followed by symbols signifying the kind of policy, as a symbol for the surplus for the nth year:

$${}_{n}B_{x} = ({}_{n-1}V_{x} + P'_{x} - \text{exp.})(1 + i') - \frac{d'_{x+n-1}}{l'_{x+n-1}}[(w\,d) - {}_{n}V_{x}] - {}_{n}V_{x}$$

This formula is general, covering both level and varying insurances, ${}_{n-1}V_{x}$ disappearing in dealing with the first year of insurance and P'_{x} disappearing in dealing with paid-up insurances.

A common variation is to apportion surplus interest according to reserves and all other gains and salvages as a percentage either upon the gross premium or upon the loading.

ELEMENTS OF SURPLUS.

It will be observed that three elements always and sometimes four elements, distinct and separable, enter into the computation of dividends, viz., salvage on mortality estimates, salvage on expense estimates or loading, gains from excess interest and accretions from forfeitures by discontinuing policies. We have obtained a formula for distributing these profits; we now need to discover just what amount of profits of each class are to be distributed.

In the case of the first element, salvage on mortality estimates, according to the old custom nothing but actual mortuary losses were included as a charge. The better practice has been to include all losses of which notice has been received, and even to allow a margin if required by the early date of computation, for losses unreported at that time. The practices of companies differ as to including expenses of adjusting claims, etc.; but the disposition to do so is increasing, and it is now considered good actuarial practice to include also examination fees and expenses of the medical department on the ground that these expenses are incurred in order to diminish mortuary losses and so are a proper deduction from the salvage to determine the net benefit. One beneficial result of this course is to prevent the mortality cost from appearing to be too low in a company's first years. The percentage of actual to expected mortality, then, may be determined as follows: % = actual losses + adjustment and medical expenses : expected losses. Or, if the percentage of aggregate net losses to aggregate costs of insurance is desired, the proposition becomes: % = actual losses — reserves + adjustment and medical expenses : aggregate costs of insurance.

These percentages will not correspond so closely as when actual losses only, *i. e.*, without adjustment and medical expenses, are dealt with, for the reason that the numerators are increased by the same amounts, not by proportionate amounts.

The amount of interest or investment gains to be divided may be determined by taking the gross gains; but such is not the approved mode nowadays. Usually investment expenses, including losses, if any, on securities or properties sold, are deducted. Whether changes in market values should be entered either as a gain or loss is yet an open question. The better view is that such changes should be noted only, first, when securities or properties are sold and the loss or gain realized; or, second, when the changes mean the cessation of income permanently or for a long time; or, third, when the changes mean a permanent loss of part or all the principal invested. Changes which signify nothing as to the revenue or ultimate value do not really affect the earnings. The formula to determine the amount of such gains thus becomes: Total investment profits realized + profits accrued —(investment expenses + investment losses).

The distribution of surplus interest offers no difficulties whatever if the full rate of interest is computed on the funds of each policy and added thereto before the reserve is deducted; but if the surplus account is kept separately, in the case of accumulative surplus it must be remembered that in addition to the surplus interest at the excess rate on the reserve and net premium, the surplus itself should be improved with interest at the full rate, the total being: excess $i\ ({}_{n-1}V_x + P_x) + i\ ({}_{n-1}\text{Accumulated Surplus}_x)$.

From the foregoing formulas it appears that two or even three sorts of expense may be dealt with other than as a charge against the loading. Eliminating these, the formula for the percentage of actual expenses to loading becomes: % = actual expenses — adjustment, medical and investment expenses : loading.

The gains from discontinuances are easily discovered, the formula being simply: Aggregate values of discontinued policies — aggregate surrender values paid. The fair distribution of these gains would be in proportion to the exposure to a similar loss. This is employed in certain tontine accumulations and also in computing annual surplus in some cases. In tontine it takes the form of apportioning these gains as a percentage on the total policy funds; in semi-tontine as a percentage on the accumulated surpluses only. Many companies inaccurately apply these gains to diminish the actual expenses before computing the percentage on the loading. This is equivalent to apportioning these gains in the ratio of the premium loadings, an unscientific proceeding, which has, however, this justification sometimes, viz., that the policy values are larger than the actual policy accumulations would be if the proper initial expense had been deducted. This is roughly equivalent to a tardy deduction.

TONTINE AND OTHER LONG-TERM DIVIDENDS.

The formula for the " contribution plan " was especially devised to cover the case of a policy with annual dividends; but with slight variations it can be applied to suit the cases of deferred dividends, whether tontine or not—that is, whether forfeited in discontinuance or not. Two forms of tontine dividends also present themselves, one with accretions from the forfeiture of the entire values of discontinued policies and the other with accretions from forfeitures of surplus only.

One manner in which the formula may be applied is to treat these dividends as essentially annual dividends, arrived at by the usual process and themselves forming a fund separate and distinct from other policy-funds which progresses by its own interest and by accretions from discontinued policies. With this fund an account is then kept in the following general form:

Former accumulations........	$——.—
New Dividend...............................	——.—
Interest..	——.—
Tontine Gains:	
Forfeitures of total values (or surpluses) of policies discontinued by lapse or surrender	——.—
Do. of surpluses of policies discontinued by death	——.—
Total	——.—

The first-mentioned item of tontine gain should, when forfeitures involve the surplus only, be ascertained by the following proportion, viz.:

Tontine gain : former accumulations + new dividend : : total forfeitures : total tontine surplus.

When forfeitures involve entire policy values, the item should be ascertained thus:

Tontine gain : reserve + former accumulations + new dividend : : total forfeitures : total reserves + total tontine surplus.

The accretions from forfeiture of surplus belonging to policies discontinued by death, which have almost facetiously been called "loss profits," present a more intricate problem for apportionment, in order to conform with the "contribution" idea. It appears, however, that the risk of forfeiture by this means is equal to the amount of the accumulation modified by the risk of death $\left(\frac{d_{x+n}}{l_{x+n}}\right)$. Whence it appears that such accretions should be divided according to the following proportion:

Tontine gain : (former accumulation + new dividend) $\times \frac{d_{x+n}}{l_{x+n}}$: : total forfeitures : total expected forfeitures.

The "expected forfeitures" should be ascertained by multiplying the total "former accumulations + new dividends" for each age by $\frac{d_{x+n}}{l_{x+n}}$ for each age and summing the results.

This formula almost accurately corresponds to what would be added to the ordinary surplus if the total forfeitures were simply deducted from the total death losses, as are the reserves, and $\frac{d'_{x+n}}{l'_{x+n}}$ computed by the following proportion:

$\frac{d'_{x+n}}{l'_{x+n}} : \frac{d_{x+n}}{l_{x+n}}$: : actual death losses — (reserve and tontine surplus) : aggregate costs of mortality.

In other words, we are proceeding as if the company paid all losses, less reserves, and then received as a forfeiture the tontine surplus, which is precisely equivalent to paying all losses, less reserves, and tontine surplus. These are mathematically equivalent, and also equivalent to a formula that charges the entire loss and then credits back on the contribution plan the entire reserve and surplus.

This leads us to a modified form of the "contribution" formula which may be used in computing deferred dividends without any resort to "accounting" or with but little resort to it. Let ${}_nF_x$ be the total fund of a policy at the beginning of the year. Then to find the amount of the fund at the end of the year:

$${}_{n+1}F_x = ({}_nF_x + P'_x - \text{exp.})\ (1 + i') - \frac{d'_{x+n}}{l'_{x+n}}[1 - ({}_nF_x + P'_x - \text{exp.})\ (1 + i')]$$

In this formula we deduct $({}_nF_x + P'_x - \text{exp.})\ (1 + i')$ from unity to get the actual insurance. This corresponds more nearly to the actual facts—death-losses being incurred and paid throughout the year instead of at the end of the year—than would the deduction of ${}_{n+1}F_x$ which would also be difficult.

The accumulated surplus is simply the excess of this fund ${}_{n+1}F_x$ over the reserve ${}_{n+1}V_x$ or, in general terms: ${}_nF_x - {}_nV_x$.

The surplus gain for the year is found by deducting the accumulated surplus at the beginning of the year from the accumulated surplus at the end of the year, thus: $({}_{n+1}F_x - {}_{n+1}V_x) - ({}_nF_x - {}_nV_x)$.

One item of tontine accretion is already covered in these formulas, viz., from forfeitures by policies discontinued by death. Their entire funds have been applied upon the death-claims. The actual insurance has been computed as unity, less the whole fund.

The other item of tontine accretion remains to be considered, viz., from forfeitures by other discontinuance.

Let us take up first the case of what are known as " full tontine " insurances which forfeit the whole fund upon discontinuance. The risk being to lose one's whole fund, the surplus is to be apportioned, according to the fund, as a percentage thereon, f′. This means that the fund ascertained by the formula already given is to be multiplied by (1 + f′), thus:

$$ {}_{n+1}F_x = (1 + f') \left\{ ({}_nF_x + P'_x - \text{exp.}) (1 + i') - \frac{d'_{x+n}}{l'_{x+n}} \right. $$
$$ \left. [1 - ({}_nF_x + P'_x - \text{exp.}) (1 + i')] \right\} $$

In the case of a policy which forfeits only the surplus, the gain is a percentage of the surplus computed by the original formula:

$$ {}_{n+1}F_x = ({}_nF_x + P'_x - \text{exp.}) (1 + i') - \frac{d'_{x+n}}{l'_{x+n}} [1 - ({}_nF_n + P'_x $$
$$ - \text{exp.}) (1 + i')]. $$

The surplus is ${}_{n+1}F_x - {}_{n+1}V_x$ and, to add the tontine gain, it is multiplied by $1 + f'$.

A third mode of computing deferred dividends is by the group system; this has been extensively employed by companies issuing tontine policies. On this plan a group of 10,000 policies, for instance, is assumed at the desired age. Columns are formed as follows:

Year.	No. Paying.	No. Dis'n'i'g.	Premiums.	Death Losses.	Expenses.	Remainders.
			$	$	$	$

In the premium column for the first year is entered the premiums on the 10,000 policies. The death-losses are next computed at the actual experience for that year; expenses the same. These are charged out of the premiums and the remainder carried to its column. The second year the premium column is credited for the number paying, *i. e.*, 10,000 less deaths and discontinuances, and this process is continued indefinitely. The remainder for the year is carried out each time. If no premiums are paid, this remainder becomes " negative " — a deduction. When the period is complete the total value is found by progressing the remainders at the actual rates of interest, thus giving the total fund, thus : increase the first remainder by interest for first year; add second remainder and increase by interest for second year, etc., to the close. This fund, divided by the number of persistent survivors, gives the amount of each individual fund.

An advantage of the group system—not confined to it, however—is that the percentages of discontinuances and of actual to expected mortality can be modified according to the policy years. Ratios by policy years are, however, only obtainable in a reliable form when the number of policies is very large. A further distinction as to discontinuances is made sometimes for different sorts of policies, as well.

REVERSIONARY DIVIDENDS APPLIED AS AN INSURANCE OR ANNUITY.

After the surplus belonging equitably to a given policy has been determined, it is a very easy matter to deal with it if the amount is payable in cash or applicable only to reduce the premiums next due. But if the dividend be applicable to increase the insurance, either temporarily or permanently or to buy an annuity, the case is of course different; an actuarial problem still confronts us.

The general principle which gives the key to the solution of each of these problems is that the cash surplus is applied as a single premium at the then age of the policyholder to buy the desired insurance or annuity.

Thus suppose it be desired to apply the ascertained cash surplus on a life policy taken at age thirty, but now five years old, to permanently increase the insurance. The amount of the reversion or addition will be determined by applying the surplus as a single premium to buy a whole life insurance at age thirty-five; thus:

$$ {}_5\text{Reversion}_{30} = \frac{{}_5B_{30}}{A'_{35}}; \text{ or, generally} $$

$$ {}_m\text{Reversion}_x = \frac{{}_mB_x}{A'_{x+m}} $$

The single premium A'_{x+m} employed in determining this reversion may be either the net premium A_{x+m} or the net premium with an especial loading or the full office premium.

The formula would be precisely the same for any whole life policy, whether single, annual, limited or renewable term, if the reversion is to be an insurance certainly payable at death.

For term and endowment insurances, where the insurance added is permanent only in the sense of persisting throughout the life of the policy, the formula changes somewhat. Suppose, for instance, a term insurance for n years on which the dividend at end of m years is to be applied to increase the insurance for the remaining term n—m; the formula becomes:

$${}_m\text{Reversion}_x = \frac{{}_mB_x}{|_{n-m}A'_{x+m}}$$

Suppose the case of an endowment insurance in n years on which the dividend at end of m years is to be applied to increase the insurance for the remaining term, but not to increase the endowment, the formula remains the same for the reason that the insurance is still term insurance valued at $|_{n-m}A'_{x+m}$.

Suppose, however, a similar case, except that the dividend is to increase both insurance and endowment, and the formula becomes:

$${}_m\text{Reversion}_x = \frac{{}_mB_x}{A'_{x+m\,\overline{n-m}|}}$$

Another form of dividend addition is that of temporary insurance, *i. e.*, an insurance not corresponding in term to the insurance of the policy. The most usual form is that of an addition for the current year only, *i. e.*, the purchase of one year term insurance at the attained age. The formula for this is:

$${}_m\text{Temp. Addition}_x = \frac{{}_mB_x}{|_1A'_{x+m}}$$

If no loading be added to the net premium, this becomes:

$${}_m\text{Temp. Addition}_x = \frac{{}_mB_x}{|_1A_{x+m}} = \frac{{}_mB_x}{\frac{v\,d_{x+m}}{l_{x+m}}} = \frac{l_{x+m}\,(\,{}_mB_x\,)(1+i)}{d_{x+m}}$$

These formulas cover the cases of applying surplus to increase the insurance; it may, however, be desired to apply the surplus to buy an annuity either for life or a term of years. The same principle applies.

Suppose the dividend at end of m years to be applied to buy a life annuity; the formula is:

Annuity $= \frac{{}_mB_x}{a'_{x+m}}$ or if the annuity desired is an immediate one, then:

$$\text{Annuity} = \frac{{}_mB_x}{1 + a'_{x+m}}$$

It sometimes occurs, as in term, limited payment and endowment policies, that the annuity is desired for the remaining years of the payment period only. Suppose such a policy with n payments of which m payments have been made, to find what annuity to reduce the premium now due and future premiums the surplus will buy; the formula becomes:

$$\text{Annuity} = \frac{{}_mB_x}{1 + |_{n-(m+1)}\, a_{x+m}}$$

PARTICIPATION OF ANNUITIES.

Annuities are nearly always issued on a non-participating basis; but there is in the nature of things no reason why they might not be issued on a participating basis, and some consideration of plans for distributing surplus thereto is proper at this place.

In life insurances there is a salvage from mortality estimates if fewer lives fail than were expected to fail; in annuities there is, on the contrary, a salvage when more lives fail than were expected to fail. Deaths in life insurances cause the payment of the insurance; deaths among annuitants stop the payment of the annuities.

So far as gains from excess interest over the hypothetical rate used in computing premiums are concerned, the cases of annuities and insurances are analogous.

Likewise so far as salvage from loading is concerned.

In the cases of both these items there is never any difficulty in apportioning the gains or salvages on the contribution plan, the excess interest according to the mean annuity fund and the salvage on the loadings according to the loadings.

A variation of the apportionment of this expense salvage would be to treat a part or all the loading as the present value of an additional annuity to be employed to defray expenses throughout the term and to apportion the salvage each year according to these additional annuities available for expenses.

But it is the ascertaining and apportioning gains from mortality salvages which offer the greatest difficulties.

The simplest method of ascertaining the amount of this salvage or loss is as follows: Improve reserve at beginning of year for a whole year and all net premiums for year for the time in hand, at assumed rate of interest; deduct reserve at end of year. Remainder is expected payments, with interest for half year; compare with actual payments plus interest for half year.

The apportionment requires a fuller explanation, during which we will investigate the effect of the mortality experience upon each annuity value.

Suppose l_x annuities at age x, the value of which is $l_x\, a_x$. During the year this amount is increased by i interest and then diminished by l_{x+1} annuity payments, and the remaining fund is the value of l_{x+1} annuities at age x + 1 or $l_{x+1}\, a_{x+1}$. But $l_{x+1} = l_x - d_x$, and it is evident that the fund has been modified by the d_x deaths in two ways: first, there have been $l_x - d_x$ annuity payments, and, second, there are only $l_x - d_x$ annuities still in force. If there were more than d_x deaths, there would be two gains: first, because less annuity payments to make, and, second, because less annuity values to provide for; and, on the contrary, if there were less than d_x deaths, there would be two losses: first, because more annuity payments to make, and, second, because more annuity values to provide for.

To return to our analysis, we have discovered that

$$l_{x+1}\, a_{x+1} = l_x\, a_x\,(1 + i) - l_{x+1}$$

$$a_{x+1} = \frac{l_x\, a_x\,(1 + i) - l_{x+1}}{l_{x+1}}$$

$$= \frac{l_x\, a_x\,(1 + i)}{l_{x+1}} - 1$$

$$= \frac{l_x}{l_x - d_x}\,[a_x\,(1 + i)\,] - 1$$

$$= \frac{1}{p_x}\, a_x\,(1 + i) - 1$$

Now, having ascertained the rate of actual interest and the percentage of gain or loss by deaths upon the values expected to be released by deaths, proceed as follows: first, make a corrected table of deaths by modifying the figures in the d column so that $\frac{d''_x}{l_x} = \frac{d'_x}{l'_x}$ for all ages. Then

$$F\,a_{x+1} = \frac{l_x}{l_x - d''_x}\Big[a_x\,(1+i')\Big] - 1.$$

To find the surplus deduct the value of the annuity:

$${}_1B\,a_x = F\,a_{x+1} - a_{x+1} \text{ or generally}$$

$${}_nB\,a_x = F\,a_{x+n} - a_{x+n}$$

SURRENDER VALUES.

HAD the first plans of insurance been scientific and the first policies mutual and participating, no doubt the reasoning about surrender values would have been about like this: The fund remaining from the premiums of a policy has not been needed to pay for insurance already enjoyed; as the policy is now to be discontinued, this fund will not be needed to pay for future insurance; therefore it may, and indeed should, be returned upon discontinuance.

Later, it is conceivable that reasons might have been discovered why some deduction from this value might be made in some cases or some addition to it in other cases, according as it was less or more desirable to have the life off the books. But it is clear that the burden of proof lies upon him who would make any change in the primary proposition which has every appearance of being just.

But the first plans of insurance were neither scientific nor mutual; and, moreover, the patrons did not understand that there was any accumulation of unearned premium. The cupidity of insurers caused them to grant grudgingly only those surrender values which, for business reasons, it seemed wisest to give. At the bottom no consideration of what ought to be given entered into the matter.

Partly under the stress of public demand, partly because of legislation or the fear of legislation and partly because of competition among companies, the various plans for computing surrender values have been evolved. In many cases they are purely arbitrary. In others they have at least the semblance of scientific accuracy.

When the subject of surrender values was first mooted by the Massachusetts commissioners, the companies even forfeited the dividend additions or reversions—insurance that was fully paid for—upon failure to pay premiums on the original policy.

The form of surrender values first proposed by the commissioners and adopted as the first law of Massachusetts on the subject was to apply the reserve to extend the insurance The premium for this term insurance might be loaded twenty-five per cent; or what was the same thing, eighty per cent of the reserve was to be so applied as a *net* single premium. We have elsewhere given formulas for computing the time of insurances, the premium and amount insured being given. At the time when the law was proposed, 1860, such a formula was not generally in use, and one objection raised against the system was the labor it involved. Elizur Wright, one of the commissioners, silenced this objection by computing net single premiums for all terms and ages and publishing the same; since then, tables of such values have been generally used. To compute extension terms from it, find first the even number of years for which the premium will pay and also the remainder of premium; then ascertain the difference between the premium for that term and a term one year longer; let Δ represent this difference and the days in excess of the even years may be ascertained by the following proportion:
Days : 365 : : remainder of premium : Δ.

This is not quite exact, as it does not take into account that the risk of death during the year is an increasing one; but nearly all ordinary actuarial computations are in this respect inaccurate.

Before this law was enacted, the New York Life Insurance Company introduced another system which has been more generally adopted, viz., paid-up insurances.

In the case of life policies payable in ten premiums, it agreed that after a certain period a paid-up policy should be granted for as many tenths of the original amount as annual premiums had been paid. This idea has been applied to all limited payment life and endowment policies. It is too simple to require exemplification.

It is also unscientific. The values granted in the earlier years are higher than the reserves would buy, and those granted in later years much lower. Some companies have always avoided this plan in spite of its simplicity and popularity. In the case of whole life insurances with continuous premiums, it is plainly inapplicable. The better and more scientific rule has been to apply the reserve to buy paid-up insurance.

Because the premiums have already contributed a full loading it has by many been considered fairer to apply the reserve as a *net* premium; but such is not the universal practice.

This is also too simple to require a formula.

The payment of cash surrender values is of yet later adoption. The determination of the amount to be paid has been a disputed matter even to this day. Against the proposition that the whole policy fund remaining unexpended should be allowed, the following objections, among others, have been interposed: it would increase the surrenders; the companies would suffer from adverse selection, those lives conscious of being in superior condition retiring and those conscious of impairment persisting at all hazards; the insured being interested in the surrender value and only the beneficiary in the ultimate proceeds, the former is tempted to betray the interests of the latter.

The last of these objections is not actuarial and the first presents no real disadvantage, even if true—and the facts are against it—unless the second be valid. That there might be adverse selection, especially if the soundness of a company is suspected, is certainly reasonable; but no evidence exists to prove that there is any considerable adverse selection in a going company with a good reputation. The insured contributes to the payment of losses other than his own, according to his age and the actual insurance, or, as we have seen, according to the "costs of insurance." Hence it follows that the effect of adverse selection on the part of those who withdraw is some function of all the "costs of insurance" to be paid by them; this, on the principle that being better lives than the average they would, if they did not discontinue, pay a little more for their insurance than the value of the benefits received. Consequently the loss by adverse selection can be offset by making a surrender charge against the reserve of a sufficient percentage of all future costs of insurance.

This was the idea which Elizur Wright introduced into the Massachusetts statutes when he caused the cash surrender law to replace that providing for extended insurance; the new law provided for a surrender charge of eight per cent of the value of the costs of insurance, called the "insurance value."

Another plan has been to allow a surrender value on the basis of some mortality or interest standard different from the one actually employed in computing the reserves. Thus, for instance, the Australian Mutual Provident Society formerly reserved on the basis of the H^{m5} table and four per cent, on policies more than five years old, but allowed as cash values the reserves on the H^{m} table with four per cent, the difference being in effect a surrender charge.

A somewhat similar plan, long employed by an American company, was based on the idea that the company should forfeit nothing of its advantage to the discontinuing policyholder. Thus this company gave as a surrender value non-participating paid-up insurance and allowed as a cash value for that a sum believed to approximate the actual cost of furnishing this insurance. To do this a mortality table nearly corresponding to the company's actual experience and a rate of interest at least equal to the actual were employed to compute the value of the paid-up insurance, which sum was allowed as a surrender value.

The first cost of procuring insurances is commonly, together with other expenses chargeable, more than the loading on first premiums. Consequently, the reserve at the end of the first year is usually not to be had from the premium itself; in other words, the actual and theoretical reserves do not agree. Nearly all companies seek to avoid paying a cash surrender value, exceeding what has been accumulated from the premiums received, after meeting actual losses and expenses. On this account some deduction from the usual reserves is commonly made in case of surrender within five years. This indicates, perhaps, that "select and ultimate" reserves would be suitable for maximum cash surrender values.

INDIVIDUAL ACCOUNTS.

Many life insurance companies keep no other individual account with a policy than a memorandum of the age, amount and kind of policy, the date of issue, frequency and amount of premiums and the fact of their payment.

The reserves on these policies they can ascertain when they will by reference to tables or by a short computation.

The surplus they ascertain at least as often as the dividend period requires, by group computations, thus making out a scale of dividends for all forms of policies at all ages.

The reserve and surplus added together at any time shows the whole fund.

For a company with a large number of policies this system is decidedly less laborious, and, since when it is faithfully carried out, it yields the same result as would any system based on individual accounts and on the same principles, it is satisfactory.

Its sole disadvantage is that it is not comprehensible by the insured, who consider the matter one of mystery in consequence.

Because of this, Elizur Wright devised a system of accounting which is readily comprehensible. It is fully set forth in his " Savings Bank Life Insurance Tables."

The idea is to divide the premiums as they are paid in into three different portions, viz., loading, cost of insurance and deposit for reserve, sending each to its own fund and crediting each individual policy with the premium and charging against it the loading and the cost of insurance.

The loading is easily ascertained by deducting the net premium from the gross.

The cost of insurance in this case is the cost at the end of the year discounted; as the amount is deducted on receipt of the premium.

When this cost is taken out of the net premium it is clear that the remainder constitutes a deposit for reserve. This, together with any previous accumulations, constitutes the net credits of the individual account, which credits are at the close of the year improved at the actual rate of interest. The salvage on the cost of mortality and the salvage on the loading for expenses are also then added and the total brought down.

The terminal reserve for that year is deducted and carried forward; and the remainder is surplus, either carried forward also as a part of the fund or withdrawn or applied.

Mr. Wright, in the book mentioned, divided the premiums for each year of many usual forms of policies at all ordinary ages of issue into these three parts, which are sometimes mistakenly called "elements." The mistake is often also made of supposing that the cost of insurance and the deposit for reserve are the same ratios all through as on the first year of insurance; instead, the line of demarcation is constantly shifting.

This system of accounting offers several advantages. It causes the company to keep its expense and mortuary provisions separate from its reserves, so that it cannot trench upon the latter without knowing it. The plan also facilitates the calculation of the saving on mortality and on loading.

Notwithstanding which facts it is somewhat intricate and involved as a method of individual accounting and subject, though in far less degree, to the same objections as the other, viz., that it is not understood by the insured.

The system is only objectionable because amounts are first charged to the individual and then some part of the same amounts credited back. The account may easily be so simplified as to be comprehensible to any business man without explanation by merely crediting the premiums, charging out the actual expenses and mortality at the end of the year and progressing the remaining fund at interest. This system also does away with the necessity for extensive tabulation. In practice, the process may be carried forward as follows: Credit the premiums; at the end of the year charge for the actual expense in proportion to the loading; charge for the mortality by computing the cost at the actual rate for the age attained, of a one year's insurance for the net actual insurance (*i. e.*, the face of the policy, less the fund remaining after expenses are deducted); credit interest on the remainder at the actual rate earned and carry forward the balance. Credit the new premium and proceed as before, charging for any dividend paid as well as for other items as before.

INSURED INSTALMENT LOANS OR ADVANCED ENDOWMENTS.

We have yet to consider the case of a company engaging, for a consideration of an annual payment for a fixed term but contingent upon the survival of the payer, to pay a sum down in advance. Of course no such engagement would be undertaken unless the annual payment were secured, which is usually done by real estate mortgage. The transaction thus becomes one of an insured instalment loan.

It is evident that this is precisely equivalent to the company's buying a temporary annuity equal to the instalment on the life of the payer.

Let S be the sum advanced, n the term and x the age. Then, to find the payment, first find the value of $|_{n}a_{x}$ and divide S by it, thus, designating the payment by π, the symbol for unusual annual premiums:

$$\pi = \frac{S}{|_{n}a_{x}} = \frac{S}{\dfrac{N_{x} - N_{x+n}}{D_{x}}} = \frac{S\,D_{x}}{N_{x} - N_{x+n}}$$

This is for the case of payments due at the end of the year; if payments are due at the beginning of the year, divide by $1 + |_{n-1}a_{x}$, the value of an immediate temporary annuity, thus:

$$\pi = \frac{S}{1 + |_{n-1}a_{x}} = \frac{S}{\dfrac{N_{x-1} - N_{x+n-1}}{D_{x}}} = \frac{S\,D_{x}}{N_{x-1} - N_{x+n-1}}$$

If the payments are to continue throughout life, a very unusual case, the payments become

$$\pi = \frac{S}{a_x} = \frac{\frac{S}{N_x}}{D_x} = \frac{S\,D_x}{N_x} \text{ and}$$

$$\pi = \frac{S}{1 + a_x} = \frac{\frac{S}{N_{x-1}}}{D_x} = \frac{S\,D_x}{N_{x-1}}, \text{ respectively.}$$

Equivalent formulas may be arrived at in another manner. If such a loan is made, it is evident that the company must collect the interest each year on the whole amount advanced and must also collect a sum equal to the net premium for an endowment insurance for the amount advanced, the maturity of the endowment insurance at death or in n years canceling the loans.

The formula for an advance of 1 thus becomes:

$$\pi = d + P_{x\,\overline{n|}} = d + \frac{M_x - M_{x+n} + D_{x+n}}{N_{x-1} - N_{x+n-1}}$$

This covers only the case of an advance with payments yearly in advance, interest also collected yearly in advance. If, however, payments are at the end of the year, the formula may be employed by altering the denominator of the fraction, thus:

$$\pi = i + \frac{M_x - M_{x+n} + D_{x+n}}{N_x - N_{x+n}}$$

These formulas will give the same results as the formulas based on considerations of a temporary annuity; which fact is but another phase of the equivalents explained in another chapter.

A variation from these last formulas is found in forms which use one rate of interest in determining the endowment premium and another in computing interest on the advance, as i and í, respectively, thus:

$$\pi = d' + P_{x\,\overline{n}|} = d' + \frac{M_x - M_{x+n} + D_{x+n}}{N_{x-1} - N_{x+n-1}}$$

when payments are yearly in advance and

$$\pi = í + \frac{M_x - M_{x+n} + D_{x+n}}{N_x - N_{x+n}}$$

when payments are at the end of the year.

The effect of this is to increase or diminish the actual rate of interest realized, according as i, the rate at which the sinking fund is progressed, is less or greater than í, the rate of interest charged on the loan. This form is, in practice, only employed to get an increased profit.

Another variation of a similar character is to add a loading to the endowment premium, as, for instance, by using the office premium, $P'_{x\,\overline{n}|}$, thus:

$$\pi = í + P'_{x\,\overline{n}|}$$

The valuation of such a contract offers no difficulties. The simplest form is to treat it as an endowment, purchased by $\pi - í$, and to find the value of the endowment which is the amount of the principal debt that has been discharged.

Where the endowment premium is net ($P_{x\,\overline{n}|}$ and not $P'_{x\,\overline{n}|}$) and at the same rate of interest as the loan, this method will give precisely the same result as a partial payment account, the actual insurance being charged for at the tabular rate of cost.

The net value of such a contract is an asset and not a liability of the company. It is the present value of the future premiums; thus, when with payments in advance:

$$\begin{aligned} {}_mV\,\pi &= (1 + |_{n-m-1}\, a_{x+m})\,\pi \\ &= \frac{N_{x+m-1} - N_{x+n-1}}{D_{x+m}} \div \frac{S\,D_x}{N_{x-1} - N_{x+n-1}} \\ &= \frac{(N_{x-1} - N_{x+n-1})\,(N_{x+m-1} - N_{x+n-1})}{S\,D_x\,D_{x+m}} \end{aligned}$$

JOINT LIFE VALUES.

The value of an endowment payable only in case x and y survive one year, is plainly:

$$E_{xy} = \frac{v\,l_{x+1}l_{y+1}}{l_x\,l_y} = \frac{v\,l_{x+1:y+1}}{l_{xy}}.$$

And:

$${}_nE_{xy} = \frac{v^n l_{x+n:y+n}}{l_{xy}}.$$

$$a_{xy} = E_{xy} + {}_2E_{xy} + {}_3E_{xy} + \ldots.$$

$$= \frac{v\,l_{x+1:y+1} + v^2 l_{x+2:y+2} + \ldots.}{l_{xy}}$$

Similarly as to insurances:

$$|\,{}_1A_{xy} = \frac{v\,(l_{xy} - l_{x+1:y+1})}{l_{xy}}$$

$$A_{xy} = \frac{v\,(l_{xy} - l_{x+1:y+1}) + v^2(l_{x+1:y+1} - l_{x+2:y+2}) + \ldots}{l_{xy}}$$

It is worthy of note that the deaths among $l_x\,l_y$ pairs is not $d_x\,d_y$ but $l_x\,l_y - l_{x+1}\,l_{y+1}$ which is the diminution in the number of pairs by the death of one or both; $d_x\,d_y$ would be the number of joint deaths only, i. e., deaths of both.

Two forms of commutation columns have been suggested:

1st. Multiply numerator and denominator of the fraction: $\dfrac{v\,l_{x+1:y+1} + v^2\,l_{x-2:y+2} + v_3\,l_{y+3:y+3} + \ldots}{l_{xy}}$ by v^x, x being the older of the two lives, so that it takes the form:

$$a_{xy} = \frac{v^{x+1}\,l_{x+1:y+1} + v^{x+2}l_{x+2:y+2} + v^{x+3}\,l_{x+3:y+3} + \ldots}{v^x\,l_{xy}}$$

$$= \frac{D_{x+1:y+1} + D_{x+2:y+2} + D_{x+3:y+3} + \ldots}{D_{xy}}$$

$$= \frac{N_{xy}}{D_{xy}}$$

Similarly in insurances, multiply numerator and denominator of the fraction:

$$\frac{v(l_{xy} - l_{x+1:y+1}) + v^2(l_{x+1:y+1} - l_{x+2:y+2}) + \dots}{l_{xy}} \text{ by } v^x,$$

x being the older of the two lives; and we have:

$$A_{xy} = \frac{v^{x+1}(l_{xy} - l_{x+1:y+1}) + v^{x+2}(l_{x+1:y+1} - l_{x+2:y+2}) + \dots}{v^x l_{xy}}$$

$$= \frac{C_{xy} + C_{x+1:y+1} + C_{x+2:y+2} + \dots}{D_{xy}}$$

$$= \frac{M_{xy}}{D_{xy}}$$

This form is known as Davies', having been invented by Griffith Davies, who introduced commutation columns into general use.

2nd. In the expression for the joint-life annuity, multiply numerator and denominator by $v^{\frac{x+y}{2}}$ which gives:

$$a_{xy} = \frac{v^{\frac{x+y}{2}+1} l_{x+1:y+1} + v^{\frac{x+y}{2}+2} l_{x+2:y+2} + \dots}{v^{\frac{x+y}{2}} l_{xy}}.$$

$$= \frac{D_{x+1:y+1} + D_{x+2:y+2} + \dots}{D_{xy}} = \frac{N_{xy}}{D_{xy}}.$$

Likewise in the expression for an insurance:

$$A_{xy} = \frac{v^{\frac{x+y}{2}+1}(l_{xy} - l_{x+1:y+1}) + v^{\frac{x+y}{2}+2}(l_{x+1:y+1} - l_{x+2:y+2}) + \dots}{v^{\frac{x+y}{2}} l_{xy}}$$

$$= \frac{C_{xy} + C_{x+1:y+1} + C_{x+2:y+2} + \dots}{D_{xy}} = \frac{M_{xy}}{D_{xy}}.$$

This second form has the advantage that the value of the commutation symbol does not depend on which life is older. It is known as De Morgan's form, having been invented by the eminent English mathematician of that name.

From commutation columns of D_{xy}, N_{xy}, C_{xy} and M_{xy}, of either form, commutation columns of S_{xy} and R_{xy} may also be made; and by means of the formulas for insurances, annuities, endowments, and reserves, for policies on one life suitably modified, all the usual joint-life values may be computed therefrom.

The most convenient and useful retrospective formula for joint-life valuation is that of Emory McClintock, which eliminates c_{xy} or k_{xy} from the computation.

Commutation columns for three or more joint lives may be made upon the same principles and the same formulas, suitably modified, will produce joint-life values for three or more lives.

When joint-life annuities for two lives only are available, roughly approximate values for three lives may be had by Simpson's rule, which is: find $a_w = a_{yz}$ (y and z being the two oldest ages) then

$$a_{xyz} = a_{xw} \text{ approximately.}$$

JOINT LIFE VALUES, EQUAL AGES AND UNIFORM SENIORITY BY MAKEHAM'S METHOD.

We have seen that when a table is constructed on Makeham's method,

$$\begin{aligned} \mu_x &= A + Bc^x \\ &= -\log_e s - (\log_e c \log_e g)c^x \end{aligned}$$

Also

$$\begin{aligned} \log l_{x+t} &= \log k + (x+t) \log s + c^{x+t} \log g \\ \log l_x \; &= \log k + x \quad \log s + c^x \log g \end{aligned}$$

$$\log {}_tp_x = \log l_{x+t} - \log l_x = t \log s + c^x (c^t - 1) \log g$$

$$\log {}_tp_y = \log l_{y+t} - \log l_y = t \log s + c^y (c^t - 1) \log g$$

$$\log {}_tp_{xy} = 2t \log s + (c^x + c^y)(c^t - 1) \log g$$

Find w such that $c^w = \frac{1}{2} (c^x + c^y)$; then:

$$\begin{aligned} \log {}_tp_{xy} &= 2t \log s + 2c^w (c^t - 1) \log g \\ &= 2 \log {}_tp_w = \log {}_tp_{ww}. \end{aligned}$$

By the differential calculus, it is readily proved that

$$\mu_{xy} = \mu_x + \mu_y = 2A + B(c^x + c^y)$$

Since $c^w = \frac{1}{2} (c^x + c^y)$, substitute and we have:

$$\mu_{xy} = 2(A + Bc^w) = 2\mu_w = \mu_{ww}.$$

$$\mu_w = \frac{1}{2} \mu_{xy} = \frac{1}{2} (\mu_x + \mu_y)$$

Therefore we may find w, by taking from a table of μ, the values μ_x and μ_y, adding these, dividing by 2, and then finding that value which most nearly corresponds to $\frac{1}{2} \mu_{xy}$. Greater accuracy may be obtained by proportioning between the ages.

Since as we have seen,

$$\log {}_tp_{xy} = \log {}_tp_{ww},$$

it follows also that $|{}_tq_{xy} = |{}_tq_{ww}$ and that other values, such as expectations, annuities, and assurances, will be the same for the joint ages x and y as for the equal joint ages, w and w

Then $a_{xy} = a_{ww}$

$A_{xy} = A_{ww}$ and so on.

When therefore a mortality table has been graduated by Makeham's method, and when a complete table of values of any function has been computed for two lives at all the equal ages, values of the same function for any two joint lives, x and y, may be quickly calculated by first finding the equal age, w, and then taking the value from the tables, for ages w and w For the sake of accuracy it will usually be desirable to interpolate for the value ; because w seldom proves to be an integer.

All that has here been said will, with appropriate changes, apply to three or more joint lives; for

$$\begin{aligned}\mu_{xyz}\ldots\ldots &= \mu_x + \mu_y + \mu_z + \ldots\ldots\\ &= \mu_w + \mu_w + \mu_w + \ldots\ldots\\ &= \mu_{www}\ldots\ldots\end{aligned}$$

This rule that when the mortality table follows Makeham's law, two equal ages may be substituted for x and y, is called the law of "uniform seniority," because when x—y = n, w—y is equal to r, a constant or uniform quantity, for all values of x and y, having a difference between the ages = n. We have:

$$\begin{aligned}c^w &= \tfrac{1}{2}(c^x + c^y)\\ &= \tfrac{1}{2}c^y(c^n + 1) = c^y\,\tfrac{1}{2}(c^n + 1)\end{aligned}$$

But $\frac{1}{2}(c^n + 1)$ is a constant quantity for all values of x and y, so long as x — y = n. Let $C = \frac{1}{2}(c^n + 1)$ and we have

$$c^w = c^y C$$

$$w \text{ Log } c = y \log c + \log C.$$

$$w = y + \frac{\log C}{\log c}$$

in which, also, $\frac{\log C}{\log c}$ = a constant quantity for all values of x and y, so long as $x - y = n$.

By means of a column of these constants for each difference of ages, the equal age w may be found without reference to the formula:

$$\mu_w = \tfrac{1}{2}(\mu_x + \mu_y).$$

For values of joint life functions at equal ages, Davies' and De Morgan's commutation formulas become identical; since $v^{\frac{w+w}{2}} = v^w$.

MAKEHAM'S LAW AND CHANGED RATE OF INTEREST.

Instead of constructing joint life tables for equal ages and deriving other joint life values therefrom as heretofore explained, it is possible when a mortality table follows Makeham's law, to construct a table of values for one life, at such changed rate of interest as would give accurate values for two joint lives. In the same way a table of values may be constructed at *another* rate of interest by which accurate values for three joint lives would be derived, and so on for other numbers of joint lives, the rate of interest being changed with relation to the number of lives involved.

The proof is as follows:

$$\text{Log } l_{x+t} = \log k + (x+t) \log s + c^{x+t} \log g.$$

$$\text{Log } l_x = \log k + x \log s + c^x \log g$$

$$\text{Log } {}_tp_x = \text{Log } l_{x+t} - \log l_x = t \log s + (c^{x+t} - c^x) \log g$$

$$= t \log s + c^x (c^t - 1) \log g$$

$${}_tp_x = s^t g^{(c^t-1)c^x}$$

Similarly

$${}_tp_{xy} = s^{2t} g^{(c^t-1)(c^x+c^y)} = s^{2t} g^{(c^t-1)c^w} = s^t s^t g^{(c^t-1)c^w}$$

and ${}_tp_{xyz\ldots\ldots(m)} = s^{mt} g^{(c^t-1)c^w} = s^{(m-1)t} s^t g^{(c^t-1)c^w}$. Where two lives are involved $c^w = c^x + c^y$; and when there are m lives $c^w = c^x + c^y + c^z \ldots\ldots (m)$.

The similarity of these expressions to

$${}_tp_x = s^t g^{(c^t-1)c^x}$$

is apparent. They take the following form to wit:

$${}_tp_{xy} = s^t \, {}_tp_w \quad .$$

$${}_tp_{xyz\ldots\ldots(m)} = s^{(m-1)t} \, {}_tp_w$$

From these, remembering that $a_{xy} = \Sigma\, v^t\, {}_tp_{xy}$, we get

(1.) $a_{xy} = \Sigma\, v^t\, s^t\, {}_tp_w$ and similarly.

(2.) $a_{xyz \ldots (m)} = \Sigma\, v^t\, {}_tp_{xyz \ldots (m)}$

$= \Sigma\, v^t\, s^{(m-1)t}\, {}_tp_w$

If in (1) we use a rate of interest, such that $v' = v\,s$ by the former rate, we have:

(3) $a_{xy} = \Sigma\, v'^t\, {}_tp_w = a'_w$

If in (2) we use a rate of interest, such that $v'' = v\, s^{(m-1)}$, we have:

(4.) $a_{xyz \ldots (m)} = \Sigma\, v''^t\, {}_tp_w = a''_w$.

PROBABILITIES OF SURVIVORSHIP.

THERE is one probability that remains to consider, viz., the probability that x will die during any given year and y be alive at the moment of his death; this is called the probability of survivorship. We give it the symbol q^{1}_{xy} or, more generally, ${}_{n-1|}q^{1}_{xy}$, which latter means the probability that x will die during the nth year, y being still alive.

This probability cleaves into two probabilities, viz., that x will die in the nth year and y survive that year, and that x will die in the nth year and y die in that year also, but after the death of x.

The first of these probabilities we know to be ${}_{n-1|}q_x\,({}_{n}p_y) = ({}_{n-1}p_x - {}_{n}p_x)\,{}_{n}p_y$.

The second of these probabilities is arrived at as follows: The probability that both x and y will die in the nth year is ${}_{n-1|}q_x\,({}_{n-1|}q_y) = ({}_{n-1}p_x - {}_{n}p_x)\,({}_{n-1}p_y - {}_{n}p_y)$. Each, on the supposition of uniform deaths throughout the year, is as likely to die in one part of the year as in another; consequently the chance that either one will die first is equal, or, in other words, $\tfrac{1}{2}\,({}_{n-1}p_x - p_x)\,({}_{n-1}p_y - {}_{n}p_y)$.

Thus the entire probability becomes:

$$
\begin{aligned}
{}_{n-1|}q^{1}_{xy} &= ({}_{n-1}p_x - {}_{n}p_x)\,{}_{n}p_y + \tfrac{1}{2}\,({}_{n-1}p_x - {}_{n}p_x)({}_{n-1}p_y - {}_{n}p_y)\\
&= ({}_{n-1}p_x - {}_{n}p_x)\,{}_{n}p_y + \tfrac{1}{2}({}_{n-1}p_x - {}_{n}p_x)\,{}_{n-1}p_y -\\
&\qquad \tfrac{1}{2}\,({}_{n-1}p_x - {}_{n}p_x)\,{}_{n}p_y\\
&= \tfrac{1}{2}\,({}_{n-1}p_x - {}_{n}p_x)\,{}_{n}p_y + \tfrac{1}{2}\,({}_{n-1}p_x - {}_{n}p_x)\,{}_{n-1}p_y\\
&= \tfrac{1}{2}\,({}_{n-1}p_x - {}_{n}p_x)\,({}_{n-1}p_y + {}_{n}p_y)\\
&= \tfrac{1}{2}\left(\frac{l_{x+n-1}}{l_x} - \frac{l_{x+n}}{l_x}\right)\left(\frac{l_{y+n-1}}{l_y} + \frac{l_{y+n}}{l_y}\right)
\end{aligned}
$$

$$= \frac{(l_{x+n-1} - l_{x+n})\,(l_{y+n-1} + l_{y+n})}{2\,l_x\,l_y}$$

$$= \frac{d_{x+n-1}\,(l_{y+n-1/2})}{l_x\,l_y}$$

When n = 1, the formula becomes:

$$q^1_{xy} = 1/2\;(\,_0p_x - p_x)\,(p_y + \,_0p_y)$$

But $_0p_x$ and $_0p_y$, the probability of x and y respectively being alive at the moment the term begins, are by hypothesis both equal to certainty or 1. Substituting, our formula becomes

$$q^1_{xy} = 1/2\;(1 - p_x)\,(1 + p_y)$$

We may further develop the formula for $_{n-1}|q^1_{xy}$ as follows:

$$_{n-1}|q^1_{xy} = 1/2(\,_{n-1}p_x - \,_np_x)\,(\,_np_y + \,_{n-1}p_y)$$

$$= 1/2\;[(\,_{n-1}p_{xy} - \,_np_{xy} - \,_np_x\,(\,_{n-1}p_y) + \,_np_y\,(\,_{n-1}p_x)]$$

But we have seen that

$$_np_x\,(\,_{n-1}p_y) = \frac{_np_{x\,:\,y-1}}{p_{y-1}}$$ and, therefore,

$$_np_y\,(\,_{n-1}p_x) = \frac{_np_{y\,:\,x-1}}{p_{x-1}},$$ as developed in the lessons on life probabilities. Substituting, we have:

$$_{n-1}|q^1_{xy} = 1/2\left(\,_{n-1}p_{xy} - \,_np_{xy} - \frac{_np_{x\,:\,y-1}}{p_{y-1}} + \frac{_np_{y\,:\,x-1}}{p_{x-1}}\right)$$

SURVIVORSHIP INSURANCE.

An insurance on the life of x which is payable only if y survive him, is called a contingent or survivorship insurance. It may be further involved by being also payable to z if y does not survive x and z does survive him, etc.; or by being a joint-life insurance on the lives of x and y, payable to z if z survives the first of them to die. Thus the number of the insured and the number of successive beneficiaries may in theory be indefinitely multiplied. In practice more than two lives are seldom dealt with and more than three lives almost never in the United States. We shall here undertake to deal with two lives only, save in cases specially mentioned.

Suppose a survivorship insurance for 1 year on the life of x, payable to y if he survive x. Remembering the formula for the probability of x dying within the year, y surviving him, *i. e.*,

$$ {}_{n-1|}q^1_{xy} = \tfrac{1}{2}\,({}_{n-1}p_x - {}_np_x)\,({}_{n-1}p_y + {}_np_y) $$

$$ q^1_{xy} = \tfrac{1}{2}\,(1 - p_x)\,(1 + p_y) $$

we have for the one year's insurance, which we will designate by the symbol $|\,{}_1A^1_{xy}$, the following formula:

$$ |\,{}_1A^1_{xy} = v\,\tfrac{1}{2}\,({}_0p_x - p_x)\,({}_0p_y + p_y) $$

$$ = v\,\tfrac{1}{2}\,(1 - p_x)\,(1 + p_y) $$

The value, now, of the second year's insurance will be $v^2\,\tfrac{1}{2}\,(p_x - {}_2p_x)\,(p_y + {}_2p_y)$ and of the nth year's insurance $v^n\,\tfrac{1}{2}\,({}_{n-1}p_x - {}_np_x)\,({}_{n-1}p_y + {}_np_y)$.

Consequently, an insurance for two years will be worth:

$$ |\,{}_2A^1_{xy} = v\,\tfrac{1}{2}\,({}_0p_x - p_x)\,({}_0p_y + p_y) $$

$$ + v^2\,\tfrac{1}{2}\,(p_x - {}_2p_x)\,(p_y + {}_2p_y) $$

And, generally:

$$
\begin{aligned}
|\,{}_{n}A^{1}_{xy} = {} & v\ \tfrac{1}{2}\,({}_{0}p_x - p_x)\,({}_{0}p_y + p_y) \\
& + v^2\ \tfrac{1}{2}\,(p_x - {}_{2}p_x)\,(p_y + {}_{2}p_y) \\
& + v^3\ \tfrac{1}{2}\,({}_{2}p_x - {}_{3}p_x)\,({}_{2}p_y + {}_{3}p_y) \\
& + \ldots\ldots\ldots\ldots\ldots\ldots \\
& + v^n\ \tfrac{1}{2}\,({}_{n-1}p_x - {}_{n}p_x)\,({}_{n-1}p_y + {}_{n}p_y)
\end{aligned}
$$

It follows that a survivorship insurance extending throughout the joint lives is worth:

$$
\begin{aligned}
A^{1}_{xy} = {} & v\ \tfrac{1}{2}\,({}_{0}p_x - p_x)\,({}_{0}p_y + p_y) \\
& + v^2\ \tfrac{1}{2}\,(p_x - {}_{2}p_x)\,(p_y + {}_{2}p_y) \\
& + v^3\ \tfrac{1}{2}\,({}_{2}p_x - {}_{3}p_x)\,({}_{2}p_y + {}_{3}p_y) \\
& + \ldots\ldots\ldots\ldots\ldots\ldots \\
= {} & {}^{\Sigma}v^n\ \tfrac{1}{2}\,({}_{n-1}p_x - {}_{n}p_x)\,({}_{n-1}p_y + {}_{n}p_y)
\end{aligned}
$$

But in considering probabilities of survivorship, we found that

$$
\begin{aligned}
|_{n-1}q^{1}_{xy} & = \tfrac{1}{2}\,({}_{n-1}p_x - {}_{n}p_x)\,({}_{n-1}p_y + {}_{n}p_y) \\
& = \tfrac{1}{2}\left({}_{n-1}p_{xy} - {}_{n}p_{xy} - \frac{{}_{n}p_{x:y-1}}{p_{y-1}} + \frac{{}_{n}p_{x-1:y}}{p_{x-1}}\right)
\end{aligned}
$$

Substituting, we have:

$$
A^{1}_{xy} = {}^{\Sigma}\ v^n\ \tfrac{1}{2}\left({}_{n-1}p_{xy} - {}_{n}p_{xy} - \frac{{}_{n}p_{v:y-1}}{p_{y-1}} + \frac{{}_{n}p_{x-1:y}}{p_{x-1}}\right)
$$

This looks much more complex than the other, but is really more readily convertible into parts which we can solve separately. First of all, the series: ${}^{\Sigma}\ v^n\ ({}_{n-1}p_{xy} - {}_{n}p_{xy}) = \Sigma v^n\,{}_{n-1}|q_{xy} = A_{xy}$. Then the series: ${}^{\Sigma}\ v^n\ {}_{n}p_{x-1:y} = a_{x-1:y}$ and ${}^{\Sigma}\ v^n\ {}_{n}p_{x:y-1} = a_{x:y-1}$.

Substituting, we have:

$$
A^{1}_{xy} = \tfrac{1}{2}\left(A_{xy} - \frac{a_{x:y-1}}{p_{y-1}} + \frac{a_{x-1:y}}{p_{x-1}}\right)
$$

CONTINUOUS INSTALMENT INSURANCE.

A plan of insurance, known as "continuous instalment insurance," has come into use in the United States, especially in recent years. It consists of an annuity certain for m years after the death of x, followed by an annuity for the after-lifetime of y.

The premiums can be computed by means of special commutation columns, as varieties of contingent or survivorship insurance premiums; but a simpler and more direct process was given by H. W. Robertson in the Transactions of the Actuarial Society of America, as follows:

A continuous instalment insurance for m years certain after death of x and for the after-lifetime of y, first payment at the end of the year in which x dies, consists of two portions, viz.:

1. An annuity certain for m years, first payment at end of year in which x dies.

2. A deferred annuity on the life of y, first payment due m years after end of year in which x dies.

The value of the first benefit is $a_{\overline{m}|}\,A_x$.

The second benefit is an annuity to y deferred m years, but with no payment the $\overline{m+1}^{\text{th}}$ year if x has survived one year, no payment the $\overline{m+2}^{\text{th}}$ year if x has survived two years, etc. In other words the value of the deferred annuity to y is diminished by an annuity for the joint life of y, m years older, and x at his present age; i. e., $A_y \frac{1}{\overline{m}|} (a_{y+m} - a_{y+m:x})$.

The whole value therefore is:

$$a_{\overline{m}|}\,A_x + A_y \frac{1}{\overline{m}|} (a_{y+m} - a_{y+m:x}).$$

For annual premiums divide $a_{\overline{m}|}\,A_x$ by an annuity depending on life of x and the other quantity by a joint life annuity for lives of x and y. The reason for using a joint annuity for the latter portion is that it has become a general practice to reduce the annual premium to the normal rate for a policy payable by m instalments from the date of death of the beneficiary if such death occur in the lifetime of the assured.

When the benefits are as under an endowment insurance, giving in event of death within n years, an annuity certain for m years and an annuity for the after-lifetime of y, and in event of survival of n years by x, an annuity certain for m years and for the after-lifetime of the last survivor of x and y, the benefits separate into:

1. An endowment insurance of $a_{\overline{m}|}$ on the life of x, due in n years or at prior death, $a_{\overline{m}|}\ A_{x\,\overline{n}|}$.

2. An annuity on the life of x, deferred m+n—1 years, ${}_{m+n-1|}a_x$.

3. An annuity on the life of y deferred m years but with no payment the $\overline{m+1}^{th}$ year if x survives one year, etc., up to the $\overline{n-1}^{th}$ year of survival of x. This may be also expressed as an annuity to y deferred m+n—1 years, but to commence earlier, namely m years after the death of x, if x die in the first n—1 years. The value is $A_{y\,\overline{m}|}^{\,1}\ (a_{y+m} - a_{y+m\,x\,\overline{n-1}|})$.

4. Less an annuity on the joint lives of x and y, deferred m+n—1 years; for the 2nd and 3rd benefits, in case x and y both survive m+n years would furnish an annuity of 2, instead of one, during their joint lifetime, ${}_{m+n-1|}a_{x\,y}$.

The entire single premium, then, becomes

$$a_{\overline{m}|}\ A_{x\,\overline{n}|} + {}_{m+n-1|}a_x + A_{y\,\overline{m}|}^{\,1}\left(a_{y+m} - a_{y+m\,x\,\overline{n-1}|}\right)$$
$$- {}_{m+n-1|}a_{x\,y}.$$

For annual premiums divide $a_{\overline{m}|}\ A_{x\,\overline{n}|}$ and ${}_{m+n-1|}a_x$ by an annuity on the life of x and the other quantities by joint-life annuities on the lives of x and y.

SURVIVORSHIP INSURANCE. SPECIAL C. AND M. COLUMNS.

ONE phase of the formula which we obtained for $_{n-1|}q^1_{xy}$ was:

$$_{n-1|}q^1_{xy} = \frac{d_{x+n-1}\ (l_{y+n-\frac{1}{2}})}{l_x\, l_y}$$

Where n = 1, this becomes:

$$q^1_{xy} = \frac{d_x\, l_{y+\frac{1}{2}}}{l_x\, l_y}$$

This is the probability that x will die during the first year, y being alive at the moment of his death. Now if 1 be then payable to y at the end of the year, its value becomes

$$|_1A^1_{xy} = \frac{v\, d_x\, l_{y+\frac{1}{2}}}{l_x\, l_y}$$

$$= \frac{v^{x+1}\, d_x\, l_{y+\frac{1}{2}}}{v^x\, l_x\, l_y}$$

Should it not be thought advisable to use the joint-life commutation columns, the calculation of survivorship insurances may be facilitated by computing C^1_{xy} and M^1_{xy} columns as follows:

$$C^1_{xy} = v^{x+1}\, d_x\, l_{y+\frac{1}{2}}$$

$$C^1_{x+1:y+1} = v^{x+2}\, d_{x+1}\, l_{y+1+\frac{1}{2}},\ \text{etc.}$$

$$M^1_{xy} = C^1_{xy} + C^1_{x+1:y+1} + \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

It is evident that

$$|_2A^1_{xy} = \frac{v^{x+1}\, d_x\, l_{y+\frac{1}{2}} + v^{x+2}\, d_{x+1}\, l_{y+1+\frac{1}{2}}}{v^x\, l_x\, l_y}$$

$$= \frac{C^{1}_{xy} + C^{1}_{x+1 . y+1}}{D_{xy}}$$

$$= \frac{M^{1}_{xy} - M^{1}_{x+2 : y+2}}{D_{xy}}.$$

Whence:

$$|_{n}A^{1}_{xy} = \frac{M^{1}_{xy} - M^{1}_{x+n : y+n}}{D_{xy}}$$

$$A^{1}_{xy} = \frac{M^{1}_{xy}}{D_{xy}}$$

For annual premiums divide by joint-life annuities.

LAST SURVIVOR AND SURVIVORSHIP ANNUITIES.

THERE may be annuity for the life of x, with succession to y for his life if he survives x, etc. Such an annuity is called an annuity for the life of the last survivor. The symbol $a_{\overline{xyz}}$ is given to this value.

The value of an annuity for the life of x and thereafter through the life of y, if he survives x, is evidently greater than a_x and less than $a_x + a_y$.

To consider it first in comparison with $a_x + a_y$ we find that it differs therefrom only in the circumstance that $a_x + a_y$ pays 2 during the joint lives of x and y, while the annuity we are considering pays but 1 or 1 less than $a_x + a_y$. But 1 for the joint lives is worth a_{xy}, therefore $a_{\overline{xy}}$ is worth a_{xy} less than $a_x + a_y$, thus:

$$a_{\overline{xy}} = a_x + a_y - a_{xy}$$

$$= \frac{N_x}{D_x} + \frac{N_y}{D_y} - \frac{N_{xy}}{D_{xy}}$$

We may consider it, second, with reference to a_x. It is greater than a_x by the value of an annuity to y during his after lifetime if he survives x. This latter annuity we may call $a_{x|y}$. We have, then, the formula:

$$a_{\overline{xy}} = a_x + a_{x|y}$$

It will be noted that the first term of these two formulas is the same quantity, viz., a_x. We have

$$a_{\overline{xy}} = a_x + a_y - a_{xy} = a_x + a_{x|y}$$

$$\therefore \quad a_{x|y} = a_y - a_{xy}$$

$$= \frac{N_y}{D_y} - \frac{N_{xy}}{D_{xy}}$$

That is: a survivorship or reversionary annuity on the life of y beginning at the death of x if y survive, equals a life annuity on the life of y, less an annuity for the joint lives of x and y.

The following is a formula for an annual premium for this survivorship annuity:

$$P\,(a_x\,|_y) = \frac{\dfrac{N_y}{D_y} - \dfrac{N_{xy}}{D_{xy}}}{\dfrac{N_{x-1\,:\,y-1}}{D_{xy}}} = \frac{N_y\,D_{xy}}{D_y\,(N_{x-1\,:\,y-1})} - \frac{N_{xy}}{N_{x-1\,:\,y-1}}$$

OTHER TEMPORARY CONTINGENT FUNCTIONS.

Yet another form of temporary contingent functions which are much more fully dealt with in the text-books generally than are continuous instalment functions, but are of little practical importance in America, will be here described, mainly in order that they may not be confused with the continuous instalment functions.

$Q^{1}_{x\,\overline{y(\overline{t|})}}$ = the chance that x will die during the lifetime of y or within t years thereafter. The $\overline{}|$ means that t is a term certain, the ( ) that it begins after the death of y and the $\overline{}$ that the last survivor of y and $(\overline{t|})$ is meant:

Its probability plainly consists of two parts, viz.:

1st. That x will die within t years; for even though y die this year, the contingency that x will die within t years is embraced; $|_t q_x$.

2nd. That x may die in the $\overline{t+1}^{th}$ year if y survive the first year, may die in the $\overline{t+2}^{th}$ year if y survive the second year, and so on. This contingency can only happen provided x has first survived t years $({}_tp_x)$ and then that y has survived 1 year, x dying in the $\overline{t+1}^{th}$ year, ${}_tp_x\,\{p_y\,(1-p_{x+t})\}$; that y has survived 2 years, x dying in the $\overline{t+2}^{th}$ year, ${}_tp_x\,\{{}_2p_y\,(p_{x+t}-{}_2p_{x+t})\}$, etc. The general term in this series, is ${}_tp_x\,\{{}_np_y\,({}_{n-1}p_{x+t}-{}_np_{x+t})\}$. There remains also the chance that y will die in a specified year and x in the t^{th} year thereafter at such a time as would be within t years of the date of the death of y. For this the general term is ${}_tp_x\,\frac{1}{2}\,\{({}_{n-1}p_y-{}_np_y)\,({}_{n-1}p_{x+t}-{}_np_{x+t})\}$.

The whole probability therefore is:

$$
\begin{aligned}
Q^{1}_{x\,\overline{y(\overline{t|})}} &= |_t q_x + \Sigma\, {}_tp_x \left\{ {}_np_y \left({}_{n-1}p_{x+t} - {}_np_{x+t} \right) \right\}. \\
&\quad + \Sigma\, {}_tp_x \tfrac{1}{2} \left\{ \left({}_{n-1}p_y - {}_np_y \right) \left({}_{n-1}p_{x+t} - {}_np_{x+t} \right) \right\} \\
&= 1 - {}_tp_x + \Sigma \tfrac{1}{2}\, {}_tp_x \left\{ \left({}_{n-1}p_y + {}_np_y \right) \left({}_{n-1}p_{x+t} - {}_np_{x+t} \right) \right. \\
&= 1 - {}_tp_x + {}_tp_x \; Q_{\overset{1}{\overline{x+t}}:y} \\
&= 1 - {}_tp_x \left\{ 1 - Q_{\overset{1}{\overline{x+t}}:y} \right\}
\end{aligned}
$$

By the same reasoning, we may find the value of $A^{1}_{x\,\overline{y(\overline{t|})}}$, the single premium for an insurance upon the life of x, payable in case he die before y or within t years thereafter. It consists of two portions:

1. A temporary insurance for t years.
2. An insurance payable if x die after t years but within t years after the death of y.

The formula is:

$$
\begin{aligned}
A^{1}_{x\,\overline{y(\overline{t|})}} &= |_tA_x + \Sigma\, v^{t+n}\, {}_tp_x \tfrac{1}{2} \left({}_{n-1}p_{x+t} - {}_np_{x+t} \right) \left({}_{n-1}p_y + {}_np_y \right) \\
&= |_tA_x + v^t\, {}_tp_x\, A_{\overset{1}{\overline{x+t}}:y} \\
&= |_tA_x + \frac{D_{x+t}}{D_x} A_{\overset{1}{\overline{x+t}}:y} \\
&= A_x - {}_{t|}A_x + \frac{D_{x+t}}{D_x} A_{\overset{1}{\overline{x+t}}:y} \\
&= A_x - \frac{D_{x+t}}{D_x} \left(A_{x+t} - A_{\overset{1}{\overline{x+t}}:y} \right).
\end{aligned}
$$

Annual premiums are computed by dividing either by a joint life annuity or an annuity payable throughout the status, viz.: the joint-life of x and of y plus t years after the death of y. The symbol for this annuity is $a_{x\,\overline{y(\overline{t|})}}$; and it, likewise, consists of two parts, viz.:

1. A temporary annuity for t years upon the life of x.
2. An annuity upon the life of x, deferred t years, the

first payment made only if y survive 1 year and x, $\overline{t+1}$ years, etc. The value is:

$$
\begin{aligned}
a_{x\,\overline{y(\overline{t}|)}} &= a_{x\,\overline{t}|} + \Sigma\, v^{n+t}\,{}_{n+t}p_x\,({}_np_y) \\
&= a_{x\,\overline{t}|} + v^t\,{}_tp_x\;\Sigma\,(v^n\,{}_np_{x+t:y}) \\
&= a_{x\,\overline{t}|} + \frac{D_{x+t}}{D_x}\,a_{x+t:y} \\
&= a_x - {}_{t|}a_x + \frac{D_{x+t}}{D_x}\,a_{x+t:y} \\
&= a_x - \frac{D_{x+t}}{D_x}\left(a_{x+t} - a_{x+t:y}\right)
\end{aligned}
$$

COMPLEX PROBLEMS IN SURVIVORSHIP.

Let ${}_{n}p^{[r]}_{\overline{w\,x\,y\,z\,\ldots\ldots\,(m)}}$ be the symbol for the probability that exactly r lives out of a group of m lives will survive n years the m lives being aged w, x, y, z . . . years respectively.

Let w, x . . . . (r) represent r specified lives; the chance that they will survive one year and that all the rest y, z . . . will fail to survive is: $p_{w\,x\,\ldots\ldots\,(r)}\,[(1-p_y)(1-p_z)$ . . . (m—r factors)]. It is apparent that for n years the form of the expression is exactly the same. Similarly for every other group of r lives which can be selected. The total probability sought is the sum of these partial probabilities, viz.:

$p_{w\,x\,\ldots\ldots\,(r)}\,[(1-p_y)(1-p_z)$ . . . . . (m—r factors)]
$p_{w\,y\,\ldots\ldots\,(r)}\,[(1-p_x)(1-p_z)$ . . . . . (m—r factors)]
$p_{w\,z\,\ldots\ldots\,(r)}\,[(1-p_x)(1-p_y)$ . . . . . (m—r factors)]
$p_{x\,y\,\ldots\ldots\,(r)}\,[(1-p_w)(1-p_z)$ . . . . . (m—r factors)]
p . . . [. . . . . . .]
etc. etc.

Let Z^r represent the sum of the total number of combinations of joint probabilities of living which can be formed out of m lives taken r at a time; and in like manner Z^{r+1} the sum of the combinations of joint probabilities of m lives taken (r+1) at a time, and so on.

Also let (m, r) be a symbol for the combinations of m things taken r at a time, that is for $\frac{\underline{|m}}{\underline{|m-r}\;\underline{|r}}$; and (m—1, r+1) the combinations of (m—1) things taken (r+1) at a time, and so on.

Then after multiplying out the various sections of the total probability above given, and bringing the results together, we get:

$$p^{[r]}_{\overline{wxyz\ldots\ldots(m)}} = Z^r - Z^{r+1}\frac{(m-r,1)(m,r)}{m,\ r+1} + Z^{r+2}\frac{(m-r,\ 2)\ (m,\ r)}{m,\ 1+2}$$

$$- Z^{r+3}\frac{(m-r,\ 3)\ (m,\ r)}{m,\ r+3} + \ldots\ldots$$

$$= Z^r - (r+1)\ Z^{r+1} + \frac{(r+1)\ (r+2)}{2}\ Z^{r+2}$$

$$- \frac{(r+1)\ (r+2)\ (r+3)}{3}\ Z^{r+3} + \ldots\ldots \qquad (1)$$

$$= Z^r\ (1+Z)^{-(r+1)} = \frac{Z^r}{(1+Z)^{r+1}}$$

The form marked (1) is the most useful. The following is an illustration of how it may be used. To find the probability that just three lives will survive one year out of five lives. In this m = 5, r = 3; we must find:

(1) How may combinations of three lives can be formed out of five lives, viz :

$$p_{xyw} + p_{xyz} + p_{xys} + p_{xwz} + p_{xws}$$
$$+ p_{xzs} + p_{ywz} + p_{yws} + p_{yzs} + p_{wzs}$$

(2) How many combinations of four lives can be formed out of five lives, viz.:

$$p_{xywz} + p_{xyzs} + p_{xwzs} + p_{ywzs}$$

(3) How many combinations of five lives can be formed from 5 lives, viz.:

$$p_{xywzs}.$$

The whole probability becomes:

$$p^{[3]}_{\overline{xyzws}} = (p_{xyw} + p_{xyz} + p_{xys} + p_{xwz} + p_{xws}$$
$$+ p_{xzs} + p_{ywz} + p_{yws} + p_{yzs} + p_{wzs})$$
$$- 4\ (p_{xywz} + p_{xyzs} + p_{xwzs} + p_{ywzs})$$
$$+ 10\ p_{xywzs}$$

The probability that at least r lives out of m will survive n years, is evidently the sum of the probabilities that exactly r lives will survive that exactly r+1 lives will survive, etc.; that is:

$$
\begin{aligned}
{}_n p_{\overline{xyz\ldots.(m)}}^{\;r} &= {}_n p_{\overline{xyz\ldots.(m)}}^{\;[r]} + {}_n p_{\overline{xyz\ldots.(m)}}^{\;[r+1]} + \ldots. \\
&= Z^r - {}_{r+1}Z^{r+1} + \frac{(r+1)\,(+2)}{2} Z^{r+2} - \ldots. \\
&\qquad + Z^{r+1} - (r+2)\, Z^{r+2} + \ldots. \\
&\qquad\qquad + Z^{r+2} - \ldots. \\
&= Z^r -{}_r Z^{r+1} + \frac{r\,(r+1)}{2} Z^{r+2} - \ldots. \qquad (2.) \\
&= Z^r (1+Z)^{-r} = \frac{Z^r}{(1+Z)^r}
\end{aligned}
$$

Therefore the probability that at least three lives out of five will survive one year, is:

$$
\begin{aligned}
p_{\overline{xyzws}}^{\;3} = {} & (p_{xyw} + p_{xyz} + p_{xys} + p_{xwz} + p_{xws} \\
& + p_{xzs} + p_{ywz} + p_{ywz} + p_{yzs} + p_{wzs}) \\
& - 3\,(p_{xywz} + p_{xyzs} + p_{xwzs} + p_{ywzs}) \\
& + 6\, p_{xywzs}
\end{aligned}
$$

The probability that one life will survive out of the five lives, becomes:

$$
\begin{aligned}
p_{\overline{xyz\ldots(m)}}^{\;1} = {} & (p_x + p_y + p_z + p_w + p_s) \\
& - (p_{xy} + p_{xz} + p_{xw} + p_{xs} + p_{yz} + p_{yw} + p_{ys} \\
& \qquad + (p_{zw} + p_{zs} + p_{ws}) \\
& + (p_{xyw} + p_{xyz} + p_{xys} + p_{xwz} + p_{xws} \\
& \quad + p_{xzs} + p_{ywz} + p_{yws} + p_{yzs} + p_{wzs}) \\
& - (p_{xywz} + p_{xyzs} + p_{xwzs} + p_{ywzs}) \\
& + p_{xywzs}.
\end{aligned}
$$

If all the m lives be of equal age and the term be n years, this will become:

$$
{}_n p_{\overline{xxx\ldots(n.)}}^{\;1} = m\,{}_n p_x - \frac{m\,(m-1)}{\underline{|2}} \left({}_n p_x\right)^2 + \frac{m\,(m-1)\,(m-2)}{\underline{|3}} \left({}_n p_x\right)^3 \ldots.
$$

By a similar process, we may find:

$$ {}_{n}p_{\overline{xxx\,.\,.\,.\,(m)}}^{[1]} = m\,{}_{n}p_{x} - m(m-1)\,({}_{n}p_{x})^{2} + \frac{m\,(m-1)\,(m-2)}{\underline{|2}}\,({}_{n}p_{x})^{3}\,.\,.\,.\,. $$

If we pass from probabilities of survival to annuities, we have:

$$ a_{\overline{wxyz\,.\,.\,.\,.\,(m)}}^{[r]} = \Sigma\, v^{n}\,{}_{n}p_{\overline{xyz\,.\,.\,.\,.\,(m)}}^{[r]} $$

$$ = Z^{r} - (r+1)\,Z^{r+1} + \frac{(r+1)(r+2)}{2}\,Z^{r+2}\,.\,.\,.\,. $$

in which Z refers to combinations of annuities, instead of probabilities.

Similarly:

$$ a_{\overline{wxyz\,.\,.\,.\,.\,(m)}}^{r} = \Sigma\, v^{n}\,{}_{n}p_{\overline{xyz\,.\,.\,.\,.\,(m)}}^{r} $$

$$ = Z^{r} - r\,Z^{r+1} + \frac{r\,(r+1)}{2}Z^{r+2}\,.\,.\,.\,. $$

$$ a_{\overline{xxx\,.\,.\,.\,.\,(m)}}^{1} = \Sigma\, v^{n}\,{}_{n}p_{\overline{xxx\,.\,.\,.\,.\,(m)}}^{1} $$

$$ = m\,a_{x} - \frac{m(m-1)}{2}\,a_{xx} + \frac{m\,(m-1)\;m-2)}{\underline{|3}}\,a_{xxx} - \,.\,.\,.\,. $$

$$ a_{\overline{xxx\,.\,.\,.\,.\,(m)}}^{[1]} = \Sigma\, v^{n}\,{}_{n}p_{\overline{xxx\,.\,.\,.\,.\,(m)}}^{[1]} $$

$$ = m\,a_{x} - m\,(m-1)\,a_{xx} + \frac{m\,(m-1)\,(m-2)}{\underline{|2}}\,a_{xxx} - \,.\,. $$

$a_{\overline{xyz\,.\,.\,.\,.\,(m)}}^{1}$ is more frequently written $a_{\overline{xyz\,.\,.\,,\,.\,(m)}}$, signifying an annuity payable until the death of the last survivor. For the value of such we have the following examples:

1 life $\quad a_{x} = a_{x}$

2 lives $\quad a_{\overline{xy}} = a_{x} + a_{y} - a_{xy}$

3 lives $a_{\overline{xyz}} = a_{x} + a_{y} + a_{z} - (a_{xy} + a_{xz} + a_{yz})$
$\qquad + a_{xyz}$

4 lives $a_{\overline{xyzw}} = a_{x} + a_{y} + a_{z} + a_{w} - (a_{xy} + a_{xz} +$
$\qquad a_{xw} + a_{yz} + a_{yw} + a_{zw}) + (a_{xyz}$
$\qquad + a_{xyw} + a_{xzw} + a_{yzw})$
$\qquad - a_{xyzw}$.

The insurance corresponding to one of these annuities is an insurance payable at the end of the year when the status which calls for the annuity payment fails; as, for instance, an insurance upon the death of the last survivor. Its value in net single premium is evidently 1, less d into an annuity due, of the sort corresponding with the insurance.

For example:

$$A_{\overline{x\,y\,z\,w}} = 1 - d\,(1 + a_{\overline{x\,y\,z\,w}}).$$

For annual premiums divide by the annuity on the right hand side of the equation; or get directly from conversion tables by the formula:

$P_{\overline{x\,y\,z\,w}} = \dfrac{1}{1 + a_{\overline{x\,y\,z\,w}}} - d$, entering the conversion tables with the function $a_{\overline{x\,y\,z\,w}}$

SURVIVORSHIP. THREE LIVES.

Let ${}_{n-1}|q^1_{xyz}$ represent the probability that x will die in the nth year, y and z surviving. It evidently consists of four portions, viz.:

1. That x will die in the nth year, y and z both surviving the nth year, namely; $({}_{n-1}p_x - {}_np_x)\,({}_np_y)\,({}_np_z)$

2. That x will die in the nth year, and y also, z surviving, x dying before y; $\tfrac{1}{2}\,({}_{n-1}p_x - {}_np_x)\,({}_{n-1}p_y - {}_np_y)\,({}_np_z)$.

3. That x will die in the nth year and z also, y surviving the year, x dying before z; $\tfrac{1}{2}\,({}_{n-1}p_x - {}_np_x)\,({}_{n-1}p_z - {}_np_z)\,{}_np_y$.

4. That x, y and z will all die in the nth year, but x first of the three; $\tfrac{1}{3}\,({}_{n-1}p_x - {}_np_x)\,({}_{n-1}p_y - {}_np_y)\,({}_{n-1}p_z - {}_np_z)$.

The total value becomes:

$$\begin{aligned}{}_{n-1}|q^1_{xyx} &= ({}_{n-1}p_x - {}_np_x)({}_np_y)({}_np_z) + \tfrac{1}{2}\{({}_{n-1}p_x - {}_np_x)({}_{n-1}p_y - {}_np_y)\\ &\quad ({}_np_z) + \tfrac{1}{2}\,({}_{n-1}p_x - {}_np_x)({}_np_y)({}_{n-1}p_z - {}_np_z) + \tfrac{1}{3}({}_{n-1}p_x - {}_np_x)\\ &\quad ({}_{n-1}p_y - {}_np_y)\,({}_{n-1}p_z - {}_np_z)\\ &= \tfrac{1}{3}\,({}_{n-1}p_{xyz} - {}_np_{xyz}) - \tfrac{1}{3}\,[{}_np_x\,({}_{n-1}p_{yz}) - {}_{n-1}p_x\\ &\quad ({}_np_{yz})] + \tfrac{1}{6}\,[{}_{n-1}p_{xy}\,({}_np_z) - {}_np_{xy}\,({}_{n-1}p_z) + {}_{n-1}p_{xz}\,({}_np_y)\\ &\quad - {}_np_{xz}\,({}_{n-1}p_y)]\end{aligned}$$

The entire probability that x will die first of the three lives, x, y and z is represented by Q^1_{xyz}, and

$$\begin{aligned}Q^1_{xyz} &= \Sigma\,{}_{n-1}|q^1_{xyz}\\ &= \tfrac{1}{3}\left\{1 - \left(\frac{e_{x:y-1:z-1}}{p_{y-1:z-1}} - \frac{e_{x-1:y:z}}{p_{x-1}}\right)\right\}\\ &\quad + \tfrac{1}{6}\left\{\frac{e_{x-1:y-1:z}}{p_{x-1:y-1}} - \frac{e_{x:y:z-1}}{p_{z-1}} + \frac{e_{x-1:y:z-1}}{p_{x-1:z-1}} - \frac{e_{x:y-1:z}}{p_{y-1}}\right\}\end{aligned}$$

In a similar manner the value of the corresponding insurance may be found:

$$A^1_{xyz} = \Sigma\, v^n\,{}_{n-1}|q^1_{xyz}$$

$$= \tfrac{1}{3}\left\{A_{xyz} - \left(\frac{a_{x:y-1:z-1}}{p_{y-1:z-1}} - \frac{a_{x-1:y:z}}{p_{x-1}}\right)\right\}$$

$$+ \tfrac{1}{6}\left\{\frac{a_{x-1:y-1:z}}{p_{x-1:y-1}} - \frac{a_{x:y:z-1}}{p_{z-1}} + \frac{a_{x-1:y:z-1}}{p_{x-1:z-1}} - \frac{a_{x:y-1:z}}{p_{y-1}}\right\}$$

Let $A^{2}_{x\underset{1}{y}z}$ be the symbol for an insurance payable upon the death of x provided it occurs after the death of y but before the death of z.

$$A^{2}_{x\underset{1}{y}z} = A^{1}_{xz} - A^{1}_{xyz}.$$

Similarly A^{2}_{xyz} represents the insurance payable on the death of x if he die second of the three lives.

$$A^{2}_{xyz} = A^{2}_{x\underset{1}{y}z} + A^{2}_{xy\underset{1}{z}}$$

$$= A^{1}_{xz} + A^{1}_{xy} - 2\,A^{1}_{xyz}$$

$$A^{3}_{xyz} = A_{x} - A^{1}_{xyz} - A^{2}_{xyz}$$

$$= A_{x} - A^{1}_{xz} - A^{1}_{xy} + A^{1}_{xyz}$$

$$A^{1}_{x\overline{yz}} = A^{1}_{xyz} + A^{2}_{xyz}$$

$$= A^{1}_{xy} + A^{1}_{xz} - A^{1}_{xyz}$$

$$A_{\overset{1}{\left|\overline{xy}\right|}:z} = A^{1}_{xyz} + A_{x\overset{1}{y}z}$$

$$A_{\overline{xy}:\underset{3}{z}} = A^{2}_{x\underset{1}{y}z} + A_{\underset{1}{x}\overset{2}{y}z}$$

$$= A^{1}_{xz} + A^{1}_{yz} - A^{1}_{xyz} - A_{x\overset{1}{y}z}$$

In obtaining annual premiums great care is required to see that the proper annuity value is employed.

MANY UNUSUAL ANNUITIES AND INSURANCES.

Otherwise puzzling problems may often be solved very simply and easily by analyzing the functions into simpler elements.

Thus we have:

$$a_{y}|_{x} = a_{x} - a_{xy}.$$

and similarly we find:

$$|_{n}a_{y}|_{x} = |_{n}a_{x} - |_{n}a_{xy}$$

$$_{n}|a_{y}|_{x} = {}_{n}|a_{x} - {}_{n}|a_{xy}$$

Or, if s be any status and z any other status

$$a_{z}|_{s} = a_{s} - a_{zs}.$$

$$a_{\overline{n}|}|_{s} = a_{s} - a_{s\overline{n}|}$$

$$a_{s}|_{\overline{n}|} = a_{\overline{n}|} - a_{s\overline{n}|}$$

This form of analysis may be used to solve many otherwise difficult problems, such as:

$$a_{z}|_{xy} = a_{xy} - a_{xyz}$$

$$a_{yz}|_{x} = a_{x} - a_{xyz}$$

$$a_{z}|_{\overline{xy}} = a_{\overline{xy}} - a_{\overline{xy}:z}$$

$$a_{\overline{yz}}|_{x} = a_{x} - a_{x:\overline{yz}}$$

$$a_{xyz}|_{abc} = a_{abc} - a_{abcxyz}$$

$$a_{\overline{xyz}}|_{abc} = a_{abc} - a_{abc:\overline{xyz}}$$

$$a_{xyz}|_{\overline{abc}} = a_{\overline{abc}} - a_{\overline{abc}:xyz}$$

$$a_{\overline{xyz}}|_{\overline{abc}} = a_{\overline{abc}} - a_{\overline{abc}:\overline{xyz}}$$

$$= {}_{\overline{abcxyz}} - a_{\overline{xyz}}$$

SICKNESS INSURANCE.

SUPPOSE in a group of m persons aged x, there are r weeks of illness in one year. Then, expressing the rate of sickness by z_x, we have:

$$z_x = \frac{r}{m}$$

If each person ill is to receive 1 during each week of his illness z_x must be contributed by each person. But if contributions are at the beginning of the year, $v^{1/2} z_x$ will suffice, since on the average sick benefits are payable at the middle of the year.

The value at age x of the second year's insurance is $E_x (v^{1/2} z_{x+1}) = \frac{D_{x+1} v^{1/2} z_{x+1}}{D_x}$; and generally the value of the nth year's insurance is $\frac{D_{x+n-1} v^{1/2} z_{x+n-1}}{D_x}$.

An insurance for two years, then, becomes: $v^{1/2} z_x + \frac{D_{x+1} v^{1/2} z_{x+1}}{D_x}$ which, for uniformity's sake, we may reduce to a common denominator, thus: $\frac{D_x v^{1/2} z_x + D_{x+1} v^{1/2} z_{x+1}}{D_x}$.

Thus, also, an insurance for the whole of life, which we will call s_x, becomes:

$$s_x = \frac{D_x v^{1/2} z_x + D_{x+1} v^{1/2} z_{x+1} + D_{x+2} v^{1/2} z_{x+2} + \ldots}{D_x}$$

The values ($D_x v^{1/2} z_x$, etc.) may be summed from each age through the table and given the symbol K_x, K_{x+1}, etc. Substituting, our formula becomes:

$$s_x = \frac{K_x}{D_x}$$

We also develop formulas for temporary and deferred insurances, as follows:

$$|_n s_x = \frac{D_x v^{1/2} z_x + D_{x+1} v^{1/2} z_{x+1} + \ . \ . \ . \ D_{x+n-1} v^{1/2} z_{x+n-1}}{D_x}$$

$$= \frac{K_x - K_{x+n}}{D_x}$$

$$_n|s_x = \frac{D_{x+n} v^{1/2} z_{x+n} + D_{x+n+1} v^{1/2} z_{x+n+1} + \ . \ . \ . \ . \ . \ . \ . \ .}{D_x}$$

$$= \frac{K_{x+n}}{D_x}$$

Annual premiums in advance are developed thus:

$$P s_x = \frac{s_x}{1 + a_x} = \frac{K_x}{D_x} \div \frac{N_{x-1}}{D_x} = \frac{K_x}{N_{x-1}}$$

$$_tP s_x = \frac{s_x}{1 + |_{t-1} a_x} = \frac{K_x}{D_x} \div \frac{N_{x-1} - N_{x+t-1}}{D_x} = \frac{K_x}{N_{x-1} - N_{x+t-1}}$$

$$P |_n s_x = \frac{|_n s_x}{1 + |_{n-1} a_x} = \frac{K_x - K_{x+n}}{D_x} \div \frac{N_{x-1} \quad N_{x+n-1}}{D_x} = \frac{K_x - K_{x+n}}{N_{x-1} - N_{x+n-1}}$$

$$P_n | s_x = \frac{_n| s_x}{1 + |_{n-1} a_x} = \frac{K_{x+n}}{D_x} \div \frac{N_{x-1} - N_{x+n-1}}{D_x}$$
$$= \frac{K_{x+n}}{N_{x-1} - N_{x+n-1}}$$

Suppose, however, as is frequently the case, the company engages for a payment of one sum during each week of the first six months of illness, then for a smaller sum each week for six additional months, and then for a smaller for the remaining duration of the illness.

Then we must make $v^{\frac{1}{2}} z_x$ = the value of the benefit for the first six months, $v^{\frac{1}{2}} z'_x$ = the value of the benefit for the second six months and $v^{\frac{1}{2}} z''$ = the value of the benefit for the remaining illness, $\frac{1}{2}$ being the average period elapsing before benefits are paid. Then we must compute $D_x v^{\frac{1}{2}} z_x$, $D_x v^{\frac{1}{2}} z'_x$ and $D_x v^{\frac{1}{2}} z''_x$ values for each age and sum them to get K_x, K'_x and K''_x columns. Then our formulas become:

$$s_x = \frac{K_x + K'_x + K''_x}{D_x}$$

$$P\, s_x = \frac{K_x + K'_x + K''_x}{N_{x-1}}, \text{ etc.}$$

Of course, such columns can be made for any combination.

Disability in old age is hardly an illness, arising, as it does, from age itself. Consequently, and also because life has little earning value in old age, it is customary to have the insurance stop at age 65 or 70 and often to have an annuity begin at that age. Suppose an illness insurance of 1 a week to age 70 and an immediate annuity of 50, beginning at 70. The value is approximately:

$$|_{70-x}\, s_x + 52({}_{69-x}|\bar{a}_x) = \frac{K_x - K_{70} + 50\, N_{69}}{D_x}$$

The annual premium is approximately:

$$P\, [|_{70-x}\, s_x + 52({}_{69-x}|\bar{a}_x)] = \frac{K_x - K_{70} + 50\, N_{69}}{N_{x-1} - N_{69}}$$

The reserves on sickness policies are computed on the same principle as life insurance reserves, viz., by deducting from the value of future benefits the value of future premiums, thus:

$$ {}_mV\,s_x = s_{x+m} = \frac{K_{x+m}}{D_{x+m}} $$

$$ {}_mV\,P\,s_x = s_{x+m} - P\,s_x\,(1 + a_{x+m}) $$

$$ = \frac{K_{x+m}}{D_{x+m}} - \frac{K_x}{N_{x-1}}\left(\frac{N_{x+m-1}}{D_{x+m}}\right) $$

$$ {}_mV_t\,P s_x = s_{x+m} - {}_tP\,s_x\,(1 + |_{t-m-1}\,a_{x+m}) $$

$$ = \frac{K_{x+m}}{D_{x+m}} - \frac{K_x}{N_{x-1} - N_{x+t-1}}\left(\frac{N_{x+m-1} - N_{x+t-1}}{D_{x+m}}\right) $$

$$ {}_mV|_n\,s_x = |_{n-m}\,s_{x+m} = \frac{K_{x+m} - K_{x+n}}{D_{x+m}} $$

$$ {}_mV\,P\,|_n\,s_x = |_{n-m}\,s_{x+m} - P\,|_n\,s_x\,(1 + |_{n-m-1}\,a_{x+m}) $$

$$ = \frac{K_{x+m} - K_{x+n}}{D_{x+m}} - \frac{K_x - K_{x+n}}{N_{x-1} - N_{x+n-1}} $$

$$ \left(\frac{N_{x+m-1} - N_{x+n-1}}{D_{x+m}}\right) $$

$$ {}_mV_n|\,s_x = {}_{n-m}|\,s_{x+m} = \frac{K_{x+n}}{D_{x+m}} $$

$$ {}_mV\,P_n|\,s_x = {}_{n-m}|\,s_{x+m} - P_n|\,s_x\,(1 + |_{n-m-1}\,a_{x+m}) $$

$$ = \frac{K_{x+n}}{D_{x+m}} - \frac{K_{x+n}}{N_{x-1} - N_{x+n-1}}\left(\frac{N_{x+m-1} - N_{x+n-1}}{D_{x+m}}\right) $$

CONVERSION TABLES.

The student must have observed that there is a certain relation between annual and single premiums and between both of these and annuities which holds whatever be the nature of the insurance. Thus:

$$A = 1 - d\,(1 + a) = 1 - vi\,(1 + a)$$

$$P = \frac{1}{1 + a} - d = \frac{1}{1 + a} - vi$$

This holds good whether the insurances be on one life or many lives, the annuity being of the same order. Thus:

$$A_x = 1 - d\,(1 + a_x)$$
$$A_{\overline{xy}} = 1 - d\,(1 + a_{\overline{xy}})$$
$$P_{x\overline{n|}} = \frac{1}{1 + {}_{n-1|}a_x} - d$$
$$A_{xy} = 1 - d\,(1 + a_{xy})$$
$$A_{xy\overline{n|}} = 1 - d\,(1 + |_{n-1}\,a_{xy})$$

Consequently, if a of a certain sort or "status" be known, we may easily find either A or P of the same status, provided we also know the rate of interest; for the other factors in the second part of the equations are known quantities. A and P, then, may be found by reference to these formulas (a and i being known), without knowing anything about the mortality table. Temporary annuities give endowment insurances.

These formulas are so simple and so easily applied that it would seem that no one would care for anything more or better. But the ingenuity and industry of the older actuaries have spared little for their successors to do in this as in many other matters.

William Orchard, in a work published in 1856, gave to the world tables from which to one-thousandth of the unit (dollar or pound) the single or annual premium may be ascertained (the annuity being known and the rate of interest). Orchard's tables give premiums corresponding to annuity values differing by hundredths of the unit, and he furnishes differences for making the values approximate even smaller fractions.

So extensive tables are hardly requisite, as, indeed, are no such tables, since computation from the formulas offers no difficulties. The Institute of Actuaries' Text-Book gives a conversion table, corresponding only to unit values of the annuity, with tables of differences corresponding to the fractions in the annuity value.

The computation of the premiums in these tables would offer no difficulties, being direct from the formulas; but the differences were determined by separate computations by a different method.

First as to single premiums: Suppose A to be known, then to find by how much A should be increased or diminished, a having been increased by Δa. We have:

$$A = 1 - d(1 + a)$$

$$A + \Delta A = 1 - d(1 + a + \Delta a)$$

Subtracting the former equation from the latter, we have

$$\Delta A = - d(\Delta a)$$

That is to say, the difference is subtractive, which means that A diminishes as a increases—a thing plainly true since a large annuity value means small decrement by deaths, which in turn means low insurance premiums.

Making $\Delta a = .1$ successively, we find that ΔA becomes a fixed quantity the moment the value of i is given. Thus when $i = 4\%$, $\Delta a = .1$, then

$$\Delta A = - .03846154 \times .1 = .003846154$$

For every .1, then, that the annuity is more than the whole number value in the table, .003846154 is subtracted from A; and for every .1 that the annuity is less than the whole number value, .003846154 is added to A. Likewise for every .01, more or less, .0003846154 is subtracted or added, respectively. The difference columns in the table are mere tabulations of these differences corresponding to differences of .1 and .01 in the annuities.

Annual premium conversion tables offer no greater difficulties, except that the differences are on quite another basis. Thus:

$$P = \frac{1}{1 + a} - d$$

$$P + \Delta P = \frac{1}{1 + a + \Delta a} - d$$

Subtracting the former equation from the latter, we have

$$\Delta P = \frac{1}{1 + a + \Delta a} - \frac{1}{1 + a}$$

That is to say, assigning $\Delta a = .1$, the differences will be

$$\Delta P = \frac{1}{1 + a + .1} - \frac{1}{1 + a}$$

It is evident, also, that this difference will be minus or subtractive because the first fraction, having the larger denominator and the same numerator, is the smaller. The series of differences constructed in this manner will vary as a varies as is seen in the table of differences accompanying the conversion table. They do not, however, involve or depend upon the rate of interest, and are applicable whatever the rate of interest.

In applying the single premium conversion table, for every difference of .01 in the annuity subtract one-tenth as much from the premium as if the difference were .1, etc.

In computing annual premiums from the conversion table a different course must be pursued. The table of differences only gives the differences proper to .1, .2, etc. It will be observed that the difference for .2 is not twice .1, but an irregular amount more. Consequently for fractions smaller than .1, one must proportion. For example: given $a = 13.842$ to find P, interest three per cent. When $a = 13$, $P = .04230$. From this subtract .00386 for the .8. If this were .9, we would subtract .00432 or .00046 more. Being .842 instead of either .8 or .9, we will subtract $\frac{42}{100}$ (.00046) $= .00019$. We thus have

P for 13		=.04230
Subtract		
For .8	.00386	
For .042	.00019	.00405
P desired		.03825

CORRESPONDENCES AND EQUIVALENTS.

THE idea of accumulation is so essential to a proper understanding of the principles of actuarial computations and at the same time in some respects so puzzling and intricate, that a few words upon the subject of equivalents are needed.

We have familiarized ourselves with the idea that the value of the insurance and the value of the premiums must be equal, *i. e.*, that the two must be equivalents. This brings forward the conception that a sum in hand and instalments to be paid in future may be equivalent, and that two sets of instalments to be paid in future may be equivalent to the same sum in hand and so to each other; a conception which may be illustrated further.

Suppose a man who wished to borrow a sum of money and provide for its repayment, including interest thereon, by a certain annual payment for a fixed time. He might do this in either of four ways:

1. He might borrow the amount of one person, the interest to be compounded and the whole sum to be paid at the close of the period; and then invest a sufficient sum annually at compound interest at the same rate to accumulate the amount required to pay the whole.

2. He might borrow the amount of one person, paying the interest annually and the principal at the end of the period; and invest annually at compound interest at the same rate a sum sufficient to accumulate an amount equal to the principal.

3. He might borrow the amount with the privilege of annual partial payments, and each year after paying the

interest on the unpaid balance reduce the principal by paying thereon the remainder of his annual sum after meeting the interest.

4. He might find a man or company willing to give him the amount in return for an annual payment of the sum for the agreed period.

These four are equivalent, and the same annual sum, applied in any way of the four, will equally well discharge the debt.

In the same way an endowment insurance is equivalent to a term insurance for the same period, and a pure endowment also for the same period, and these in turn are equivalent to the premiums progressed through the period at interest, less the expected expense and mortuary costs. There is no difference in its real nature, whatever way you compute it; and reasoning based on one equivalent which does not result the same when based on any other equivalent, is necessarily misleading.

These remarks are called forth because the issue has been raised that, although these ways are equivalent, yet, somehow, the insurance value is bigger when you look at it one way than when you compute it in the ordinary way. The student should beware of being confused and misled by fallacies.

COMPARATIVE NOTATION.

In this work, except where the fact that the notations are otherwise is noted, the notation throughout is that of the Institute of Actuaries' Text-Book, adopted in 1895 by the International Congress of Actuaries as the universal standard.

The Actuarial Society of America has also adopted the same notations, with the following exceptions: Since all American books and tables have printed the commutation column, N, and formulas based thereon on the basis that $D_x + D_{x+1} + D_{x+2} + \; . \; . \; . \; . \; . \; . \; . \; . = N_x$ instead of $D_{x+1} + D_{x+2} + D_{x+3} + \; . \; . \; . \; . \; . \; . \; . = N_x$, as in the universal notation, the society adopts the symbol $\mathbf{N}_x$, printed in heavy-faced type, as equivalent to N_{x-1} of the universal system. The society also adopts $S =$ sums insured as an alternative for (w d) and $\Pi =$ single premium as an alternative for A; the expressions K_x, K_x and u_x are all distinctively American.

The adoption of this notation by the Actuarial Society of America was, in 1890, and it has hitherto been employed in no American book or tables, excepting in those papers and tables published by members of the society and mostly included in the society's publications. The following is a partial comparison of the English symbols with the symbols formerly used in America and contained in almost all American books and tables, so far as they differ:

UNIVERSAL STANDARD NOTATION.	CORRESPONDING AMERICAN SYMBOLS.
Symbols for Premiums.	
A	Π
P	π
$1 + a_x$ or $\mathbf{a}_x$	A_x
$1 + \|_{n-1}a_x$	A_{xn}
$\|_{n-1}a_x$	$A_{x}{}^{1}{}_{n}$
${}_nE_x$	${}_n\Pi_{xn}$
${}_nA\ E_x$ or $A_{x\overline{n\|}}$	${}_e\Pi_{xn}$
$P_n\ E_x$	${}_n\pi_{xn}$
$P_{x\overline{n\|}}$	${}_e\pi_{xn}$ or e π_{xn}
${}_tP_x$	π_{xt}
$\|_nP_x$ or $P^{1}_{x\overline{n\|}}$	$t\,\pi_{xn}$
$\|_nA_x$ or $A^{1}_{x\overline{n\|}}$	Π_{xn}
${}_tP_{x\overline{n\|}}$	${}_e\pi_{x}{}^{m}{}_{n}$
Symbols for Commutation Columns.	
N_{x-1} or $\mathbf{N}_x$	N_x or $\mathbf{N}_x$
N_x or $\mathbf{N}_{x+1}$	N_{x+1} or $\mathbf{N}_{x+1}$
$N_{x\overline{n\|}}$	${}_nN_x$
$M_{x\overline{n\|}}$	${}_nM_x$
Symbols for Valuation.	
V	H
${}_nV_x$	H_{x+n}
${}_nV_t\ P_x$	$l\ H_{x+n}$
${}_nV\,\|_t\ P_x$	$t\ H_{x+n}$
${}_nV\ P_{x\overline{t\|}}$	$e\ H_{x+}{}^{m}{}_{n}$
${}_nV_s\ P_{x\overline{t\|}}$	$ze\ H_{x+}{}^{m}{}_{n}$
${}_nA\ K\ A_x$	${}_{x+n}I_x$

SHORT METHODS OF COMPUTATION.

From the mathematical viewpoint actuarial science consists mainly in " short cuts." The commutation and valuation columns are illustrations of this. Short methods of computation are therefore welcomed.

Subtraction is the sole elementary calculation that in no case admits of shortening.

In addition, considerable is gained by carrying hundreds, at least, automatically on the fingers; this greatly relieves the memory, and one soon gets so he can pick out hundreds very readily. Thus two columns may also be carried forward nearly as easily as one.

Carrying smaller totals, as twenties or forties, in this manner may also be practiced with the result of a great gain in speed. One learns to pick out these smaller totals more readily.

Multiplication may be greatly shortened where the product is only desired to a certain number of figures, as, for instance, extended to a certain number of decimals.

Let us analyze the ordinary multiplication. Take, for instance, 684.7869 × .3182, with product desired to be extended to three decimals only. We have:

```
    684.7869
       .3182
    --------
    13695738
   54782952
   6847869
 20543607
 -----------
217.89919158
```

Of this product the last five figures are not needed. In practice the first foreshortening is not to add any beyond the fourth decimals and to add them, mentally only, to find what to carry into the third decimals.

But we may shorten the process much more by not making the unnecessary figures; thus:

$$\begin{array}{r} 684.7869 \\ .3182 \\ \hline 137 \\ 5478 \\ 6848 \\ 205436 \\ \hline 217.899 \end{array}$$

We do this, knowing that 680 × .0002 gives us three decimals; 684 × .008 gives also three decimals, etc. In each case we are careful to carry one where the figure in the multiplier multiplied by the figure in the multiplicand next right of the one we start with indicates that we should carry.

This short system may be varied thus for convenience:

$$\begin{array}{r} 684.7869 \\ 2813 \\ \hline 205436 \\ 6848 \\ 5478 \\ 137 \\ \hline 217.899 \end{array}$$

This consists simply in discovering which figure in the multiplicand furthest toward the right needs to be multiplied by the figure in the multiplier furthest to the right to give the required number of decimals. Then set down this figure there and reverse your multiplier, writing the figures to the right instead of the left of this figure. Then multiply in the ordinary way, setting down the result as in foregoing illustrations, and beginning in each case by multiplying the figure by the one immediately over it in the multiplicand.

There is also a short system of division based on the same idea. Let us first consider the usual form:

```
.3182 | 217.899 | 684.786 +
        19092
        -----
         26997
         25456
         -----
         15230
         12728
         -----
          25020
          22274
          -----
           27460
           25456
           -----
            20040
            19092
            -----
              948
```

Now the adding of all these ciphers seems of little use. Suppose we shorten it, thus:

$$
\begin{array}{r|r|l}
\underline{.3182} & 217.899 & \underline{684.786+}\\
 & \underline{19092} & \\
 & 26979 & \\
 & \underline{25456} & \\
 & 1523 & \\
 & \underline{1273} & \\
 & 250 & \\
 & \underline{223} & \\
 & 27 & \\
 & \underline{25} & \\
 & 2 &
\end{array}
$$

In this example, after we reach the second figure in the quotient, we divide each successive time with a diminishing divisor, thus: .318, .32, .3, taking care, however, to watch for "carries" in multiplying back.

When dealing with very large numbers, multiplication and division become very formidable operations. This is especially true when numbers are being raised to powers, as "squared," "cubed," etc. No means of shortening these operations has been devised that compare for speed and accuracy with the use of logarithms.

No attempt will be made in this book to explain logarithms. Text-books on their computation and use are many. Their use, moreover, is easily explained even though the user does not understand the wherefore. Suffice it here to say that by their use multiplication is performed by a single addition and division by a single subtraction. Raising to powers is also performed by a single multiplication and extracting a root by a single division. The most formidable computations become simple by their use.

A new short system of multiplication has been recently made known by Prof. D. N. Lehmer of the University of California.

It is illustrated, as follows:

$$\begin{array}{l} 3.1415 \\ 2.7183 \\ \hline 6.18382515 \\ 2.3557143 \\ \hline 8.53953945 \end{array}$$

The work is mental and is as follows in detail:

$$\begin{array}{l} 3 \cdot 2 = 6 \\ 3 \cdot 7 + 1 \cdot 2 = 23 \\ 3 \cdot 1 + 1 \cdot 7 + 4 \cdot 2 = 18 \\ 3 \cdot 8 + 1 \cdot 1 + 4 \cdot 7 + 1 \cdot 2 = 55 \\ 3 \cdot 3 + 1 \cdot 8 + 4 \cdot 1 + 1 \cdot 7 + 5 \cdot 2 = 38 \\ 1 \cdot 3 + 4 \cdot 8 + 1 \cdot 1 + 5 \cdot 7 = 71 \\ 4 \cdot 3 + 1 \cdot 8 + 5 \cdot 1 = 25 \\ 1 \cdot 3 + 5 \cdot 8 = 43 \\ 5 \cdot 3 = 15 \end{array}$$

It could have been written down:

$$\begin{array}{l} 6 \\ 23 \\ 18 \\ 55 \\ 38 \\ 71 \\ 25 \\ 43 \\ 15 \\ \hline 8.53953945 \end{array}$$

But the alternating spaces in the upper and lower lines give an opportunity to set down the figures neatly and conveniently in two lines.

The foregoing is not really a contracted process, except in setting down the figures. The entire multiplication is performed. But it may be contracted thus, five figures only being desired:

$$\begin{array}{r} 3.1415 \\ 2.7183 \\ \hline 6.1838 \\ 2.3557 \\ \hline 8.5395 \end{array}$$

The process of multiplication would then stop at product 71, of which the 7 would be taken only.

The following new short system of division was invented by Emory McClintock and has not been published before. It is now given with his permission.

Since the proportion

$$\frac{a}{b} = \frac{x}{1}$$

is the foundation of division, if we reduce b to 1 and affect a in the same manner, we shall obtain x. The reduction of b, the divisor, to 1 followed by not more than five ciphers, is usually easily done by adding or subtracting functions of b or of the successive results; the same operations are then easily performed on a, which should give a correct to four places and approximate in the fifth quotient.

Thus to divide 134488 by 96432.

	96432		134488
Add .03⅓	3214	Add .03⅓	4483
	99646		138971
Add .003⅓	321	Add .003⅓	448
	99967		139419
Add .0003⅓	33	Add .0003⅓	46
	100000		139465

QUESTIONS AND EXAMPLES.

INTEREST, SIMPLE.

1. The rate of interest, the amount of the principal and the amount of the accumulation being given, derive from first principles a formula for the term. Show how this is changed when the term is given and the rate of interest is to be found.

2. Prove that $d = v\,i = \frac{i}{1 + i}$.

INTEREST, COMPOUND.

1. The rate of interest, the amount of the principal and the amount of the accumulation being given, derive from first principles a formula for the term. Show how the formula is altered when the principal is 1.

2. The term, the amount of the principal and the amount of the accumulation being given, derive a formula for the rate of interest. Show how the formula is altered when the principal is 1.

3. The amount of the principal, the amount of the accumulation and the nominal rate of interest payable m times per annum being given, derive a formula for the term. Show how the formula is altered when the principal is 1.

4. The term, the amount of the principal and the amount of the accumulation being given, derive a formula for the nominal rate of interest payable m times per annum. Show how the formula is altered when the principal is 1.

5. Prove a formula for the effective rate of interest per annum in terms of the nominal rate.

6. Prove the formula for δ ; for $-\delta$.

7. Prove that $\delta = \frac{i + d}{2}$ approximately and give the correction to three terms.

8. Prove an exact formula and derive therefrom the approximate formula for the term of years in which the principal sum is doubled by compound interest.

9. Find the effective rate of interest when the nominal rate, convertible monthly, is 7%. Find the nominal rate convertible quarterly when the effective rate is 4%.

10. $400, accumulated at 3½%, convertible quarterly, amounts to $650. Find the term.

11. Calculate to 6 places of decimals the value of δ when $i = 3\%$.

12. Find the difference between the present value of $100 due in 10 years, at simple interest and at compound interest at 5% per annum, convertible quarterly.

13. When the term is 15 years, the principal $1000 and the amount is $1816.70, find the rate of interest (a) convertible annually; (b) convertible quarterly.

14. When the principal is $1500, the nominal rate of interest convertible monthly is 5%, and the term 11¼ years, find the amount.

15. In how many years does money double itself at 7% interest, convertible annually ?

16. Find the simple and compound interest on $150 for one month @ 3%, convertible yearly, to three places of decimals.

17. $40, invested 25 years ago at interest, convertible semi-annually, now amounts to $160; what is the rate ?

18. What is the meaning of δ ? Calculate its value when $i = .04$.

ANNUITIES CERTAIN. SIMPLE INTEREST.

1. Prove that $a_{\infty} = \frac{1}{i}$.

2. Derive the formulas for $s_{\overline{n|}}$ and for $a_{\overline{n|}}$ at simple interest.

3. Explain the various approximate formulas for $a_{\overline{n|}}$.

4. Compute by the various approximate formulas, the present value of an annuity of $50 for 20 years at 4% simple interest.

ANNUITIES CERTAIN. COMPOUND INTEREST.

1. Derive the formulas for $s_{\overline{n|}}$ and $a_{\overline{n|}}$.

2. Derive the formulas for $s^{(p)}_{\overline{n|}}$ and $a^{(p)}_{\overline{n|}}$. Show how modified when annuity accrues momently.

3. Interest being payable m times per annum, derive formulas for $s_{\overline{n|}}$ and $a_{\overline{n|}}$, $s^{(p)}_{\overline{n|}}$ and $a^{(p)}_{\overline{n|}}$, $s^{(m)}_{\overline{n|}}$, and $a^{(m)}_{n}$, $\bar{s}_{\overline{n|}}$ and $\bar{a}_{\overline{m}}$; and from these derive similar formulas, interest being payable momently.

4. Prove expressions for a_{∞}, $a^{(p)}_{\infty}$ and $\bar{a}_{\infty}$.

5. Derive the formula for the present value of an annuity increasing in arithmetical progression, and show how altered when a perpetuity. Also show to what expressions these formulas reduce, when the annuity is of 1, increasing 1 per annum.

6. Derive the formula for the present value of an annuity, increasing in geometrical progression. Explain how altered when a perpetuity, (a) when $r < 1 + i$ and (b) when $r > 1 + i$.

7. Give an expression for a_{n} in terms of $s_{\overline{n|}}$ with reasons therefor.

8. Given that $v = .970874$ and $v^{20} = .553676$, compute (a) the present value and (b) the amount, of $15 payable at the end of each year for 20 years.

9. Prove that the sum of the first n terms of a table of amounts of annuities certain $(s_{\overline{n}|})$ = the $\overline{n+1}$th term less $(n + 1)$ and divided by the rate of interest. Given that $v^{20} = .456387$, find the sum of the first 20 values of $s_{\overline{n}|}$ at 4%.

10. If an annuity for a certain number of years is worth p years' purchase and an annuity for double the number of years q years' purchase, find the rate of interest.

11. Find the present value of an annuity certain of 1 for n years and interpret its meaning when n becomes $n + {}^1/_m$.

12. Write down the formula for the present value of an annuity certain for n years, instalments payable k and interest m times a year; and show how the expression will be modified when (1) $m = \infty$, (2) $k = \infty$, (3) $m = k = \infty$.

13. Given that $a_{\overline{20}|} = 13.590$, calculate $a^{(2)}_{\overline{20}|}$, interest in both cases 4%, convertible annually.

14. Find the shortest time in which an annuity of x payable annually in advance $= x + y$ at least (a) without interest (b) with interest, convertible annually.

15. Find the yield of consols, paying 2½% on £100, in semi annual instalments, in perpetuity, the purchase price being 97.

16. Find the value at 3%, convertible annually, of a feu duty (perpetual rent) of £20 per annum, with a duplicand (double-duty) each 20th year, the first being due in 10 years.

17. State and explain under what circumstances the present value of a perpetuity increasing in geometrical progression may be found.

18. Find the value of an annuity certain whose payments are 1, 2, 3, &c. (a) for n years, (b) forever.

19. What is the present value at 4% per annum of an annuity commencing at $10 and increasing $2 each year (a) payable for 20 years (b) payable forever?

20. What is the present value at 4% per annum of an annuity of $5 increasing one-fifth each year and payable for 20 years?

21. Give a formula for determining approximately the rate of interest in an annuity certain. Apply it to find the rate when $a_{\overline{20}|}$ $= 14.462$.

BONDS, DEBENTURES, INSTALMENT LOANS.

1. The principal of a bond or debenture, the present value of the principal, the rate of interest yielded and the rate of interest to be realized being given, derive the formula for the present value. Give the alternative formula and explain it; and derive from it the formula for the premium.

2. Derive the formula for the total present value of a bond or debenture, (a), the rate of interest payable and the rate used in valuation both being taken as paid p times per annum; (b), the rate of interest payable only being so taken.

3. Derive the formula for the total present value of consols, (a) paying interest annually, (b) paying interest p times per annum, the rate used in valuation being larger than the rate yielded in both cases.

4. For an instalment loan, the principal being paid in equal annual instalments of $\frac{1}{m}$ each year, derive formulas for the n^{th} interest payment and the amount of the outstanding principal after the n^{th} payment has been made. Name the limitation of these formulas.

5. Upon a loan, to be repaid principal and interest by annual instalments of 1 for n years, derive formulas for (a) the first payment upon the principal; (b) the m^{th} payment on the principal and the amount remaining unpaid, after the m^{th} payment; (c), the like formulas, interest being at the same effective rate, but the instalments being $^1/_p$, p times per annum.

6. Explain the alternative viewpoints of a loan, repayable principal and interest by equal periodical instalments; show their equivalence and point out the conditions under which they cease to be equivalent.

7. Write down formulas for the amount of annual instalment to be paid for n years for a loan of 1, (a), to return interest at rate i'

on the net actual investment; (b), to return rate i′ on the total sum originally invested, reinvestment being at i only; (c), to return rate i′ on the total sum originally invested, compounding at rate i′ throughout the period, re-investment being at rate i only.

8. Derive an expression for the value of $a_{\overline{n}|}^{(i+i')}$; and show the amount required after m years to redeem the loan (a) if at request of payer; (b), if at request of payee; and (c) if by mutual desire.

9. Upon a loan of $10,000, repayable principal and interest at 5% convertible semi-annually, by 50 equal instalments at the end of each six months (a), compute the amount of each instalment and (b) show how much of each of the first four instalments is interest and how much is repayment of principal.

10. On a loan of $1000, repayable principal and interest by 10 instalments, one at the end of each year, to yield lender 5%, reinvestment being at 3%, compute (a) the annual instalment and (b) the redemption price after 5 years, at borrower's request. Given that $(1.03)^5 = 1.1593$ $(1.03)^{10} = 1.3439$ $(1.05)^5 = 1.2763$ and $(1.05)^{10} = 1.6289$.

11. Find the price at which a bond for $1000, repayable in 10 years and bearing 3%, payable annually, will yield 5%.

12. The purchaser of an annuity certain for n years wishes to obtain interest at rate i′ upon his investment; he can reinvest only at rate i. Find the annuity he requires for each unit invested. If his investment is $1000, the term 20 years, $i' = .05$ and $i = .035$, find the annuity and state what portion of each of the first four payments is interest and what portion, repayment of principal.

13. Which is the better investment, a 3½% perpetuity bought at 114 or a 3% bond, redeemable at par in 15 years, bought at 99?

14. A loan of S is to be repaid by n equal semi-annual payments of p, covering principal and interest, the first instalment due in two months. Find a formula for the semi-annual instalment and give the first three payments on the sinking fund.

15. Which yields the better return, a 4% perpetuity, bought @ 133 or a 3½% bond, repayable at par in 20 years, bought @ 108?

16. For a loan of $100, repayable principal and interest @ 4%, by six equal semi-annual instalments, find the amount of each payment and apportion each into interest and repayment of principal. Given, $(1.02)^{6} = .888$.

17. Given $a_{\overline{25}|}$ @ 4% $= 15.622$ and $a_{\overline{10}|} = 8.111$, find the repayment of principal in the 15th instalment of an annuity for 25 years.

18. A loan of 2np was granted x years ago, interest at i semi-annually on outstanding balances, principle repayable p at end of each six months for n years, find the redemption price at rate i′.

19. A loan of $10,000 at 4%, convertible semi-annually, repayable principal and interest by 40 equal instalments at end of each six months, find (a) the amount of each instalment (b) the amount of repayment of principal in the 1st and in the 21st instalment and (c) the amount of principal repaid and the amount outstanding after the payment of the 20th instalment. Given, $(1.02)^{20} = .69297$.

20. Find approximately the rate of interest realized on a bond of $1000, interest 4% payable semi-annually, redeemable at end of 30 years @ 110, and bought for 114. Given $a_{\overline{60}|}$ @ 1¾% $= 36.964$.

21. Find the present value of an annuity of $20 for 20 years, so that the purchaser may realize 6%, reinvestment being @ 3%. What is the redemption price at end of 10 years (a) when the purchaser wishes to realize (b) when the payer wishes to redeem and (c) when by mutual desire ?

22. State how to find the rate of interest realized by buying at k premium a bond, paying at rate i and redeemable at par in n years.

PROBABILITIES.

SIMPLE PROBLEMS IN RATE-MAKING.

1. If a thing happens in *a* ways and fails to happen in *b* ways, derive formulas for the probability (*a*) that it will happen, (b) that it will fail to happen and (c) for each in terms of the other.

2. Show by reasoning that $p^{1,\,2\text{ and }3} = (p^1)\,(p^2)\,(p^3)$ and derive formula for $q^{1,\,2\text{ or }3}$.

3. Derive a formula for the probability that one thing or another thing may happen (a) when they are mutually exclusive; (b) when the second may happen only in case the first has already happened.

4. Deduce a correct formula for a single premium for three years and show the error in the usual commutation into twice the annual premium.

PROBABILITIES OF MORTALITY AND SURVIVAL.

MORTALITY TABLES.

1. Deduce formulas for l_{x+1}, for p_x and for q_x. Give the formula for q_x in terms of l, only.

2. Give a formula for m_x and derive the formulas for p_x and q_x in terms of m_x. Indicate the uses for these formulas.

3. By the graphic method, graduate the following unadjusted values of q_x :

AGE.	q_x	AGE.	q_x
20	.00326	30	.01123
21	.01014	31	.00634
22	.00578	32	.00871
23	.00189	33	.00532
24	.00714	34	.00918
25	.00784	35	.01072
26	.00683	36	.00814
27	.00416	37	.01248
28	.00893	38	.00791
29	.00387	39	.01124

4. Graduate by the graphic method the following unadjusted values of l_x :

AGE.	l_x	AGE.	l_x
35	.83714	45	.76684
36	.83048	46	.76584
37	.82628	47	.75878
38	.82002	48	.75012
39	.81187	49	.74163
40	.80274	50	.73045
41	.79726	51	.71912
42	.78834	52	.70974
43	.77914	53	.69812
44	.77203	54	.68693

Give reasons why graduation by q_x is preferable.

5. Deduce the formula for Finlaison's Method of Graduation.

6. Apply Finlaison's method to graduate the following unadjusted values of p_x :

AGE.	p_x	AGE.	p_x
25	.99504	35	.92624
26	.99103	36	.91987
27	.98416	37	.91111
28	.97684	38	.90428
29	.97103	39	.89016
30	.96218	40	.88412
31	.95629	41	.87250
32	.94918	42	.86517
33	.94117	43	.85623
34	.93328	44	.84422

7. Describe De Moivre's, Gompertz's and Makeham's laws of mortality. Give expressions for l_{x+n} by each.

8. Deduce formulas for values of c, $\log_e g$ and $\log_e k$ by Gompertz's formula, and show how to construct a table of values $\log l_x$.

9. Deduce formulas for values of c, $\log_e s$, $\log_e g$ and $\log_e k$ by Makeham's formula, and show how to construct a table of values of $\log l_x$.

10. Define "force of mortality" and give the common approximate formula for the value of μ_x, showing its similarity to an expression for the value of m_x.

11. Give approximate value of m_x in terms of μ_x; in constants (a) by Gompertz's law (b) by Makeham's law.

12. Write down the expression for the value u_x by finite differences and deduce Woolhouse's expression for the value of l_x, in finite differences.

13. Derive Woolhouse's formula for graduation and simplify by increasing the radix.

14. Write down Ackland's process for applying Woolhouse's formula. Apply it to graduate the unadjusted values of l_x given in question 4.

15. Prove the correctness of Ackland's process.

16. What is a select table? An ultimate table? Name specimens of each.

17. Illustrate and explain (a) Dr. Sprague's method of arranging the values of a select table and (b) the method employed in publishing the British experience tables (1863-93).

18. Explain the requirements for a good graduation and the modes of testing the same. Test the graduations made under questions 4 and 14.

19. Write down expressions for $_np_x$, $_np_{xy}$, $_{|n}q_x$, $_{|n}q_{\overline{xy}}$, $_np_{\dot{x}y}$, $_{|n}q_{xy}$, $_{n-1|}q_x$, and $_{n-1|}q_{xy}$. Explain the reason for each.

20. Show that the probability that x will survive n years and y survive n — 1 years $= \frac{_np_{x:y-1}}{p_{y-1}}$.

21. From the American Experience Table, find p_{40} and q_{40}. Derive m_{40} (a) from p_{40}, (b) from q_{40}.

22. **Find μ_{45} American Experience Table.**

23. **Find ${}_{10}p_{[40]}$ by Select Table. Also ${}_{3}|q_{[35]}$.**

24. **What is (a) the probability that the first life of x and y will fail in the nth year (b) that the second life will fail in the nth year, (c) that one only of the lives will fail in the nth year, (d) that both will fail in the nth year, (e) that neither will fail in the nth year, (f) that one at least will fail in the nth year, (g) that x will survive n years and y, (n — 1) years?**

ENDOWMENTS AND LIFE ANNUITIES.

1. **Derive formulas for E_x, ${}_nE_x$, $|_{n}a_x$ and a_x.**

2. **Show how a complete table of values of a_x may be successively deduced.**

3. **Develop the formulas for D_x and N_x and give formulas for E_x, ${}_nE_x$, $|_{n}a_x$ and a_x in commutation symbols.**

4. **Deduce a formula for ${}_{n|}a_x$. Show the relations of $|_{n}a_x$ and ${}_{n|}a_x$ to a_x and to one another.**

5. **Compute E_{30}, ${}_{15}E_{45}$, $|_{10}a_{30}$, and a_{55}, American Experience Table, 3½%.**

6. **Calculate by a continued process the last 15 values of a_x, American Experience 3%.**

7. **Compute a_{30}, $|_{15}a_{30}$ and ${}_{15|}a_{30}$ and prove their correctness.**

8. **Compute D_x and N_x for ages 90 to end of table, American Experience, 2½%.**

9. **Compute ${}_{10}|_{15}a_{40}$, American Experience, 3½%.**

LIFE INSURANCE AND ENDOWMENT INSURANCE SINGLE PREMIUMS.

1. **Give what you consider the best definition of "premium.' Name some others, with criticisms.**

2. **Find expression for A_x in terms of two life annuities.**

3. Develop formulas for D_x and M_x and give a formula for A_x in commutation symbols.

4. Deduce formulas for $|_n A_x$ and ${}_n|A_x$ and show their relations to A_x and to one another.

5. Derive the formulas: $A_x = \frac{1 - ia_x}{1 + i} = v - da_x = 1 - d(1 + a_x) = \frac{a_x - a_x}{1 + a_x}$; and deduce the corresponding formulas for a_x.

6. Give formulas for $|_n A_x$ and $|_n Æ_x$ in terms of temporary annuities; explain each.

7. Deduce a formula for $|_n Æ_x$ in commutation symbols. And a formula for a semi-endowment insurance.

8. Demonstrate a formula for a temporary insurance, with return of net single premium in event of death.

9. Demonstrate a formula for a whole life insurance, with return of net single premium at death.

10. Demonstrate formulas for an endowment insurance with return of net single premium (a) in event of death (b) on survival and (c) both.

11. Demonstrate a formula for a pure endowment with return of the gross single premium (a) in event of death (b) on survival and (c) both.

12. Demonstrate a formula for a whole life insurance with return of the gross single premium at death.

13. Derive a formula for $(\mathrm{v}a)_x$ and deduce a formula for $(\mathrm{I}a)_x$.

14. Derive a formula for $(\mathrm{vA})_x$ and deduce a formula for $(\mathrm{I}a)_x$.

15. Derive a formula for $(\mathrm{v}_{\overline{n|}}a)_x$ and deduce a formula for $(\mathrm{I}_{\overline{n|}}a)_x$.

16. Explain the distinction between $(\mathrm{v}_{\overline{n|}}a)_x$ and $(\mathrm{v}a)_{x\,\overline{n|}}$ and between $(\mathrm{I}_{\overline{n|}}a)_x$ and $(\mathrm{I}a)_{x\,\overline{n|}}$. Derive a formula for $(\mathrm{v}a)_{x\,\overline{n|}}$ and deduce a formula for $(\mathrm{I}a)_{x\,\overline{n|}}$.

17. Give $a_x = 17.985$ and $v = .956938$, find A_x.

18. Compute D_x, C_x and M_x, age 85 to end of table, American Experience 2½%; and from these compute values of A_x at the same ages.

19. Compute A_{40}, $|_{10}A_{40}$ and $_{10|}A_{40}$. Prove.

20. Given $a_x = 14.858$ and $i = .04$, find A_x. Given $a_x = 18.558$ and $d = .038462$, find A_x. Given $a_\infty = 33.333$ and $a_x = 17.560$, find A_x

21. Given $A_x = 330.94$ and $i = .03$, find a_x. Given $A_x = 214.85$ and $d = .038462$, find a_x. Given $A_x = 269.79$ and $a_\infty = 25$, find a_x.

22. Compute $|_{15}Æ_{30}$.

23. Compute the net single premium for temporary insurance, age 35, for 20 years, with return of net single premium.

24. Compute the net single premium for a semi-endowment insurance for 25 years, age 30.

25. Compute the net single premium, age 30, for a whole life insurance with return of the net single premium at death.

26. Compute the net single premium for an endowment insurance, age 25, with return of net single premium in event of death and on survival.

27. The loading being $2.50 per $1000 insured and 10% of the net premium, calculate the net single premium for a pure endowment, age 25, due on survival of 20 years, with return of gross single premium in event of prior death.

28. The loading being $1.00 per $1000 insured and 25% of the net premium, calculate the net single premium for a whole life insurance, with return of gross single premium at death.

29. Compute the net single premium for an annuity of $15 (a) increasing $3.00 per annum; (b) decreasing $3.00 per annum. What limitation upon the latter? What is the meaning and what the value if there is no such limitation? Do you know any instance of such?

30. Compute the net single premium for $(Ia)_{30}$, $(IA)_{30}$, $(I_{\overline{10|}}\, a)_{30}$ and $(Ia)_{30\,\overline{10|}}$.

LIFE AND ENDOWMENT INSURANCE, ANNUAL PREMIUMS, ONE LIFE.

1. Write down formulas for P_x in terms of (a) A_x and a_x, (b) of d and A_x, (c) of d and a_x. Give A_x in terms of P and d; and a_x in the same terms.

2. From each of the formulas called for in question 1, develop a formula for P_x in commutation symbols.

3. Derive a formula for ${}_tP_x$ in terms of A and a and deduce a formula in commutation symbols.

4. Derive formulas for $P^{1}_{x\overline{n|}}$, $P_{x\overline{n|}}$ and ${}_nP_{n|}A_x$.

5. Develop formula for approximate value of $a_x^{(m)}$. Deduce formulas for $a_x^{(2)}$, $a_x^{(4)}$ and $\bar{a}_x$.

6. Derive formula for approximate value of ${}_{1/t|}a_x$.

7. Give the common rules in use in America for computing quarterly and semi-annual instalments of premium; and the rules to increase the net premium enough to cover the expected loss in interest only.

8. Give and explain formulas for approximate values of true quarterly and semi-annual net premiums for whole life insurances.

9. Develop a formula for a net annual premium π, for a whole life insurance, to be increased or decreased by k every t years for m periods and then to be constant at $\pi \pm mk$. Mention the limitations.

10. What is meant by $\bar{A}_x$? Give an approximate formula for its value. What adjustment must be made when applying it to endowment insurances?

11. Give the usual formula for $\mathring{a}_x$ and derive a more nearly correct formula.

12. Prove that $a_x < a_{\overline{e_x|}}$ for all values of x.

13. Describe the usual modes of loading annual premiums.

14. Discuss the purposes of loading. Show the effect on first year and renewal premiums if loading provides for expenses as incurred.

15. Derive a formula for a net annual premium for whole life insurance, with return of all net annual premiums paid, at death.

16. Derive a formula for a net annual premium for temporary insurance, with return of all net annual premiums paid, in event of death.

17. Derive a formula for a net annual premium for an endowment insurance, with return of all net annual premiums paid (a) in event of death, (b) on survival and (c) both.

18. Derive a formula for a net annual premium for t years, for a whole life insurance, with return of all net annual premiums paid, in event of death, (a) at any time (b) in t years only.

19. Derive a formula for a net annual premium for t years, endowment insurance in n years, with return of all net annual premiums paid (a) in event of death during n years, (b) on survival and (c) both. Show the changes in the first formula if premiums are returnable only in event of death during t years.

20. Derive a formula for a net annual premium whole life insurance, with all gross annual premiums returned at death.

21. Demonstrate the formula for the resultant gross annual premium (a) when a net annual extra premium is added to secure the return of all premiums in event of death within n years; (b) when a gross annual extra premium is added for the same purpose.

22. Illustrate the method of approximating the term, the age at entry and amount of net single premium being given, (a) by commutation values, (b) by tables of single premiums for temporary insurance.

23. Given $A_x = 214.85$ and $a_x = 19.414$, calculate P_x. Given $A_x = 269.79$ and $i = .04$, calculate P_x.

24. Compute P_{25}, P_{45}, P_{65} by commutation tables.

25. Given $A_x = 870.55$ and $a_{x\,\overline{19}|} = 12.5261$, compute ${}_{20}P_x$. Compute ${}_{20}P_{45}$ by commutation columns.

26. Compute $P_{30:\overline{30}|}$, $P^{1}_{30:\overline{30}|}$ and $P_{30:\overset{1}{\overline{30}|}}$.

27. Compute P_{40}, $P^{1}_{40\overline{10}|}$ and ${}_{10}P_{10|}A_{40}$.

28. Given $P_x = v - \frac{a_x}{1 + a_x}$ and $P_x = \frac{1}{1 + a_x} - d$, if term annuities for n — 1 years are substituted for a_x, what do these formulas then represent ?

29. What do the following expressions represent.

(a) $$\frac{v\,(1 + |_{n-1}\,a_x) - |_n a_x}{1 + |_{n-1}\,a_x}$$

(b) $$\frac{v\,(1 + |_{n-1}\,a_x) - |_{n-1}\,a_x}{1 + |_{n-1}\,a_x}.$$

Explain fully.

30. Given that $a_x = 20.1{:}9$, $P_x = .01825$ and $p_x = .99257$, find A_{x+1}.

31. Given values for P_x , P_{x+1} and i , find p_x .

32. Compute approximate value of $a^{(4)}_{25}$. Compute (a) usual quarterly office premiums in United States, age 25, loading 30% and (b) corresponding *true* quarterly net premiums and net instalments of the annual premium.

33. Compute for a whole life insurance of $1000, age 25, the net annual premium, to be diminished $1.00 per annum (a) for ten years and be thereafter constant, (b) until extinguished and (c) without limitation. Explain the meaning of the last.

34. Compute $\bar{A}_{40}$ and $\bar{A}_{40:\overline{15}|}$ approximately.

35. What is a complete annuity ? Compute a complete annuity, age 35, approximately. Is it more or less than the true value ?

36. Load the annual whole life premium, age 35, by $10.00 per $1000 for initial expense, a constant of $1.00 per $1000 and 20% upon the net premium. If expenses are met as they are incurred, show the result upon first year's net premium and upon the net premiums for subsequent years.

37. Compute the net annual premium at age 55 for a whole life insurance with the return at death of all net annual premiums paid.

38. Compute the net annual premium at age 30, for an insurance for 10 years, with return of all net annual premiums paid, in event of death during the period. Prove by computing net extra premium for premium return, separately.

39. Compute the net annual premium at age 30, for an endowment insurance for 20 years, with return of all net annual premiums paid (a) in event of death during the period, (b) on survival, (c) both.

40. Compute the net annual premium, payable for 10 years only, for a whole life insurance, with return of all net annual premiums paid (a) in event of death within 10 years (b) in event of death within 20 years. Prove by computing net extra premiums required. Show changes when gross annual premiums, loaded $2.00 per $1000 and 25% of net premium, are to be returned.

41. Compute $A_{[25]}$ Select 3½% and A_{25} American Experience 3½% and account for the difference (a) by explanation, (b) by demonstration.

42. Compute $P_{[\overset{1}{30}]\overline{5|}}$ Select 3½% and $P_{\overset{1}{30}\,\overline{5|}}$ American Experience and account for the difference (a) by explanation (b) by demonstration. Also $P_{[30]\,\overline{5|}}$ and $P_{30:\overline{5|}}$.

43. When $A_{\overset{1}{35}\,\overline{n|}} = 107.25$ approximate the term, (American Experience 3½%). Same when $P_{\overset{1}{35}\,\overline{n|}} = 10.18$; and same when $P_{35\,\overline{n|}} = 53.14$.

44. When $A_{30} = .401254$ and $P_{30} = .01952$, find the rate of interest.

45. Given a complete table of values of (1) a_x, (2) A_x and (3) P_x, how would you proceed to deduce the mortality table in each case.

46. (a) Show that the value of a life annuity payable m times a year, first payment due in $\frac{1}{2m}$th year, approximates $a_x + \frac{1}{2}$, whatever the value of m. (b) Find a formula for the value of an insurance payable one month after death.

RESERVES AND VALUATION.

1. Explain the nature of life insurance reserves and the unabridged forms of company balance-sheets.

2. Explain clearly (a) the distinction between a net and a gross valuation and (b) what assumptions as to loading and as to expenses and contingencies, are implied by net valuation.

3. Give a formula for ${}_nV_x$ in terms of the single premium, the annual premium and the life annuity, and explain its meaning.

4. Demonstrate a formula for ${}_nV_x$ in terms of life annuities alone.

5. Give a formula for ${}_nV_x$ in terms of annual premiums and a life annuity and explain.

6. Give a formula for ${}_mVP_{x\,\overline{n|}}$ and explain.

7. Demonstrate Wright's Accumulation Formula and show how modified for monthly values.

8. Demonstrate Fackler's formula.

9. Demonstrate McClintock's formula.

10. Write down the formula for non-continuous retrospective reserves (a) for insurance of a constant amount (b) with return of premiums. Explain the symbols.

11. Show that ${}_nV_x = v\,[q_{x+n} + p_{x+n}({}_{n+1}V_x)] - P_x$.

12. Show that ${}_nV_x = 1 - (P_x + d)\,(1 + a_{x+n})$.

13. Show that ${}_nV_x = A_{x+n}\left(1 - \frac{P_x}{P_{x+n}}\right)$.

14. From the formula for ${}_nV_x$ in terms of the life annuities, deduce a formula (a) in terms of A, (b) in terms of P.

15. If the loading on a whole life premium is $\frac{\left(P_x + \frac{1}{a_x}\right)}{.875} - P_x$ and the loading $\frac{1}{a_x}$ is for initial expense, investigate a formula for the true terminal reserve.

16. If the whole life premium is loaded as above and the limited premium for m years is the exact mathematical equivalent of the whole life gross premium, investigate a formula for the nth terminal reserve on the basis that initial expenses are made good at the outset and that all whole life loading beyond the payment period is reserved, (a) when $n < m$ and (b) when $n > m$.

17. Write down a formula for $_{n+k}V_t P_x$ in which n is integral and k fractional, and show what approximate values of A and a may be used. Show what these approximate expressions become when $k = \frac{1}{2}$.

18. Give the usual approximate formula for $_{n-\frac{1}{2}}V_x$ in terms of the terminal reserves and net annual premiums.

19. Explain the principles of the select and ultimate formula, and write down the same (a) for terminal reserve (b) for mean reserve (c) for initial reserve first year. Explain the significance of the expression which the last formula brings out.

20. Write down the formula for the nth mean reserve for a whole life policy, first year term.

21. Derive a formula for an extra net premium for each year, added to $P^{1}_{x\overline{1|}}$ the first year and to P_{x+1} thereafter, to accumulate, together with $_{n-1}V_{x+1}$ to 1 in m years. Give the formula for the nth reserve on such a policy (a) terminal (b) mean.

22. When $P_x = 19.91$, $A_{x+n} = 508.49$ and $a_{x+n} = 13.5346$, find $_nV_x$.

23. Compute the 10th terminal reserve for a whole life insurance, issued at age 40.

24. When $a_x = 18.6054$ and $a_{x+n} = 15.0867$, find $_nV_x$.

25. When $P_x = 15.10$, $P_{x+n} = 23.50$ and $a_{x+n} = 16.4461$, find $_nV_x$.

26. When $P_{x\,\overline{n|}} = 31.50$, $A_{x+n:\,\overline{n-m|}} = 578.75$ and $_{|n-m}a_{x+m} = 12.4570$, find $_mV\, P_{x\,\overline{n|}}$.

27. Compute the 7th terminal reserve of an endowment insurance for 20 years, age 25.

28. Compute c_{20} as used in Wright's Accumulation Formula and k_{20} as used in Fackler's Formula. Explain the difference and demonstrate the same.

29. Compute the first four terminal reserves, whole life, age 20, by (a) the prospective formula (b) Wright's formula (c) Fackler's formula (d) McClintock's formula, (e) the non-continuous retrospective formula. Explain the advantages and uses of each formula

30. Explain the meaning of $\frac{N_{x-n}}{D_x}$, $\frac{N_x}{D_{x+n}}$, $\frac{N_{x-n} - N_x}{D_x}$, $\frac{N_x - N_{x+n}}{D_{x+n}}$, $\frac{M_{x-n}}{D_x}$, $\frac{M_x}{D_{x+n}}$, $\frac{M_{x-n} - M_x}{D_x}$ and $\frac{M_x - M_{x-n}}{D_{x-n}}$. Also of $\frac{R_x - R_{x+n} - n\,M_{x+n}}{D_{x+n}}$.

31. Given that ${}_nV_x = 164.80$, $v = .966184$, $q_{x+n} = .010829$ and $P_x = 17.19$, derive ${}_{n+1}V_x$.

32. Given that $P_x = 44.18$, $i = .035$ and $a_{x+n} = 8.2193$, find ${}_nV_x$.

33. Given that $P_x = 34.99$, $P_{x+n} = 74.65$ and $A_{x+n} = 688.24$, find ${}_nV_x$.

34. When the whole life premium at age 35 is loaded $15 per $1000 insured for initial expense, find the first four true terminal reserves, American Experience, 3½%.

35. When the whole life premium is loaded 25% and the entire whole life loading after the limited premium period is to be reserved on a 20-payment policy, compute the 20th terminal reserve, age 35. Give an expression for the extra reserve the 14th year (a) based on retrospective principles (b) on prospective. Compute these values and prove.

36. Compute the 2nd mean reserve on an endowment policy for 20 years, age 40, (a) by the prospective formula (b) by the usual method.

37. Compute the first five terminal reserves on a whole life insurance, age 35, by the select and ultimate formula, Select 3½%. Also the first five mean reserves.

38. Explain why the differences between ultimate and select and ultimate reserves are less on limited-payment and endowment insurances than on whole life insurances at the same age. Demonstrate mathematically, also; and illustrate by first and second terminal reserves, age 35, (a) whole life, (b) 20-premium and (c) 20-year endowment.

39. Compute the net extra premium each year, age 35, to enlarge a whole life policy, first year term, (a) into an endowment insurance in 20 years, (b) into a whole life insurance paid-up in 20 years, (c) into the latter with 25% loading on whole life $(x + 1)$ net premium reserved beyond premium period.

40. Give a verbal interpretation of (a) $A_{x+n} - P_x a_{x+n}$, (b) $(P_x - P_{x+n}) a_{x+n}$, (c) $1 - (P_x + d) a_{x+n}$ and $A_{x+n} \left(1 - \frac{P_x}{P_{x+n}}\right)$.

41. Demonstrate that the initial reserve, improved at interest, equals the terminal reserve plus the value of the risk of death during the year.

COST OF INSURANCE. EXPECTED DEATHS. DIVISION AND APPLICATION OF SURPLUS. INSURED INSTALMENT LOANS.

1. Derive a formula for the cost of insurance nth year, whole life insurance.

2. Give a formula for exposed to risk at age $x + \frac{1}{2}$ and explain how to compute the expected deaths.

3. Give the formula for the contribution plan of dividing surplus. Explain.

4. Explain the methods of dividing tontine surplus. Name the one you prefer and give reasons.

5. Write down formula for actual accumulation of the fund of a life annuity. Show how to compute the surplus.

6. Explain the modes of determining surrender charges. Name your preference and give reasons.

7. Explain and illustrate the "savings bank" system of Elizur Wright, also the alternative account system.

8. Develop a formula for a payment at the beginning of each year for n years to pay off a loan of 1 in n years or at prior death; and give an alternative formula, with an explanation of the same.

9. Compute the cost of insurance the 15th year on a policy of $1000, issued at age 30, the terminal reserve being $112.34.

10. Given that the survivors within age 35, determined by nearest age at entry plus curtate duration, were on December 31st last 2134; that 815 were admitted this year at age 34, and 783 at age 35, and that there were 235 withdrawals and 20 deaths within age 35; compute the exposed to risk and the actual death rate.

11. Given that $a_x = 17.6138$, $a_{x+1} = 17.3949$ and that the actual mortality at x has been 76% of the expected for one year and the actual rate of interest realized 4.7%, compute the annuitant's pro rata share of the gain or loss for the year.

12. A policyholder writes· "The premium for my whole life policy of $1000 was $30 per annum; for my 20-year endowment, $45. Why is not my surrender value on the former $\frac{30}{45} = \frac{2}{3}$ of $1000 at the end of the 20th year?" Explain wherein this is fallacious.

13. A debt of $1000 is to be discharged by equal annual payments in advance for 20 years or to the prior death of the debtor, interest being at 4%. Compute (a) the net annual payment required, age 30 and (b) the amount of principal remaining unpaid after five years.

14. Make the necessary computations for a "savings bank" account of reserves, etc., on a 20-premium life insurance, age 35, for five years. Set out the same in a schedule.

15. Apply surplus of $42.18 to increase the amount payable (a) in event of death during the period, (b) on maturity and (c) both, under an endowment insurance for 20 years issued at age 30, 14 years having elapsed. Compute the amounts.

16. Apply surplus of $8.23 to reduce the premiums (a) for life, (b) for five years only, on a life policy, issued at age 25, 20 years elapsed.

JOINT LIFE FORMULAS.

1. Develop and explain both Davies' and DeMorgan's forms of joint life commutation columns. Which are most convenient for use, when columns for one life are at hand, and why?

2 Give Simpson's Rule for approximating the value of a joint life annuity, three lives.

3. Prove that when a table follows Makeham's law, ${}_tp_{xy}$ for all ages may be represented by ${}_tp_{ww}$.

4. Prove that when a table follows Makeham's law, the law of uniform seniority applies.

5. Prove that when a table follows Makeham's law, joint life values are equivalent to single life values at a changed rate of interest.

6. Compute the value of $a_{25 \cdot 35 \; 45}$, Hunter's Makehamized American Experience Table, 3½%, by the equal age formula; and also by Simpson's rule.

SURVIVORSHIP AND LAST SURVIVORSHIP PROBLEMS. CONTINUOUS INSTALMENTS. SICKNESS INSURANCE.

1. Demonstrate the formulas for ${}_{n-1|}q^{1}_{xy}$.

2. Demonstrate the formulas for A^{1}_{xy}.

3. Develop the formula for a continuous instalment insurance, payable 1 at the end of each year for m years after the death of x and as many years longer as y survives.

4. Develop the formula for the annual premium for a continuous instalment endowment insurance, payable 1 at the end of each year after n years or the prior death of x, for m years and as many years longer as the last survivor of x and y survives.

5. Explain the construction of commutation columns C^{1}_{xy} and M^{1}_{xy}.

6. Explain the signification of $a_{\overline{xy}}$ and $a_{x/y}$ and give formulas for their values.

7. Explain the signification of $Q^1_{x\,\overline{y(\overline{t|})}}$, $A^1_{x\,\overline{y(\overline{t|})}}$ and $a_{x\,\overline{y(\overline{t|})}}$ and give formulas for their values.

8. Demonstrate the formula for $p_{\overline{w\,x\,y\,z\,\ldots\ldots\,(m)}}^{[r]}$ and explain its signification and use.

9. Demonstrate the formula for $p_{\overline{w\,x\,y\,z\,\ldots\,(m)}}^{r}$ and explain its signification and use.

10. Explain the signification of $a_{\overline{w\,x\,y\,z\;.\;(m)}}^{[r]}$ and $a_{\overline{w\,x\,y\,z\,\ldots\,(m)}}^{r}$; give formulas for their values.

11. Explain the signification of $A_{\overline{x\,y\,z\,w}}$ and $a_{\overline{x\,y\,z\,w}}$ and give formulas for their values.

12. Explain the signification of ${}_{n-1|}q^1_{x\,y\,z}$, $Q^1_{x\,y\,z}$ and $A^1_{x\,y\,z}$ and give formulas for their values.

13 Explain the signification of $A^2_{x\,y_1\,z}$, $A^2_{x\,y\,z}$, $A^3_{x\,y\,z}$, $A^1_{x\,\overline{y\,z}}$, $A^{1}_{\overline{x\,y}\,:\,z}$, $A_{\overline{x\,y}\,.\,\underset{2}{z}}$; and give formulas for their values.

14. Explain the signification of $a_{z|\overline{x\,y}}$, $a_{\overline{y\,z}|x}$, $a_{x\,y\,z|\overline{a\,b\,c}}$; and give formulas for their values.

15 Explain the signification of s_x, $P_n|s_x$ and Ps_x ; and give formulas for their values.

16. Compute the value of ${}_{9|}q^{1}_{30\,\cdot\,40}$.

17. Compute the value of $A^{1}_{35\,\cdot\,40}$.

18. Compute the net single and annual premiums for a whole life insurance on a life aged 30, payable in continuous instalments of $100 at the end of each year for 20 years and as many years longer as the beneficiary, also aged 30, shall survive.

19. Compute the net single and annual premiums for an endowment insurance for 20 years on a life aged 35, payable in continuous instalments of $100 at the beginning of each year for 20 years after death or maturity and as many years thereafter as the last survivor of the insured and the beneficiary, aged 25, shall survive.

20. Compute the value of $a_{\overline{30:30}}$ and $a_{30|30}$ and the annual premium to secure the latter.

21. Compute the value of $A^{1}_{25\,40\,(\overline{10})}$ using value obtained in answer to 17, in the computation. Compute the net annual premium (a) payable only for the joint lives (b) payable throughout the status.

22. Compute $p^{[2]}_{\overline{25:30:35:40}}$ and $a^{[2]}_{\overline{25:30:35:40}}$.

23. Compute $p^{2}_{\overline{25:30:35:40}}$ and $a^{2}_{\overline{25:30:35:40}}$.

24. Compute $A_{\overline{25:30:35:40}}$ and $P_{\overline{25:30:35:40}}$.

25. Compute the annual premium to age 70, to furnish sickness insurance of 1 for the first 26 weeks and ¼ for each week thereafter to age 70, British Government Tables and 3½%.

CONVERSION FORMULAS. N_x AND $\mathfrak{N}_x$. CONTRACTED PROCESSES.

1. Give the conversion formula for single premiums and develop the formula for the difference. Show how to use a conversion table.

2. Give the conversion formula for annual premiums and develop the formula for the difference. Show how to use a conversion table.

3. Describe the four equivalent modes of dealing with the repayment of an instalment loan.

4. Explain the usual contracted method of multiplication. What are the advantages and disadvantages of reversing the multiplier.

5. Explain the usual contracted method of division.

6. Explain Prof. Lehmer's system of contracted multiplication.

7. Explain McClintock's system of contracted division.

8. Given $A_x = 410.03$, $i = .035$ and $\triangle a = .744$, compute A_{x+1}.

9. Given $P_x = 17.19$, $a_x = 18.6354$ and $\triangle a_x = 8148$, find P_{x+1}.

10. Compute the net single premium at 3½%, corresponding to $a = 13.5346$, by reference to a conversion table. Show process.

11 Compute the net annual premium at 3½%, corresponding to $a = 15.0867$, by reference to a conversion table. Show process.

12. Illustrate by computations the four equivalent methods of dealing with a loan of $1000, repayable principal and interest by equal instalments at the end of each six months for four years, interest at 5%. Demonstrate by each method the principal remaining due, after five instalments have been paid.

13. Multiply 8215684 by 9.278 (a) by the usual method of contracted multiplication, discarding decimals (b) by Prof. Lehmer's method. Also 14215612.81411 by 915.612917, discarding all decimals below .01.

14. Divide 94612813 by 71583 by contracted division (a) usual method (b) McClintock's method, discarding all decimals below .1 in the quotient.

AMERICAN EXPERIENCE TABLE.

(HUNTER S MAKEHAMIZED.)

Age.	l_x	d_x	p_x	q_x	Age	l_x	d_x	p_x	q_x
10	100,081	766	.99235	.00765	55	64.383	1,196	.98142	.01858
11	99.315	762	.99233	.00767	56	63.187	1,251	.98020	.01980
12	98.553	757	.99232	.00768	57	61,936	1,310	.97835	.02115
13	97.796	752	.99231	.00769	58	60,626	1,373	.97735	.02265
14	97.044	748	.99229	.00771	59	59.253	1,441	.97568	.02432
15	96,296	744	.99227	.00773	60	57.812	1,512	.97385	.02615
16	95.552	740	.99226	.00774	61	56.300	1,589	.97178	.02822
17	94,812	736	.99224	.00776	62	54.711	1,667	.96953	.03047
18	94.076	733	.99221	.00779	63	53.044	1,750	.96701	.03299
19	93.343	729	.99219	.00781	64	51,294	1,835	.96423	.03577
20	92,614	726	.99216	.00784	65	49.459	1,923	.96112	.03888
21	91,888	723	.99213	.00787	66	47.536	2,010	.95772	.04228
22	91,165	721	.99209	.00791	67	45.526	2,097	.95394	.04606
23	90,444	718	.99206	.00794	68	43.429	2,183	.94973	.05027
24	89,726	716	.99202	.00798	69	41,246	2,264	.94511	.05489
25	89,010	715	.99197	.00803	70	38.982	2,340	.93997	.06003
26	88,295	714	.99191	.00809	71	35,642	2,407	.93431	.06569
27	87,581	713	.99186	.00814	72	34.235	2,462	.92809	.07191
28	86,868	712	.99180	.00820	73	31.773	2,504	.92119	.07881
29	86,156	714	.99171	.00829	74	29,269	2,530	.91356	.08644
30	85,442	714	.99164	.00836	75	26,739	2,535	.90519	.09481
31	84,728	715	.99156	.00844	76	24,204	2,517	.89601	.10399
32	84,013	718	.99145	.00855	77	21,687	2,476	.88583	.11417
33	83,295	720	.99136	.00864	78	19,211	2,406	.87476	.12524
34	82,575	725	.99122	.00878	79	16,805	2,310	.86254	.13746
35	81,850	729	.99109	.00891	80	14.495	2,186	.84919	.15081
36	81,121	734	.99095	.00905	81	12,309	2,036	.83459	16541
37	80,387	741	.99078	.00922	82	10,273	1,863	.81805	.18135
38	79.646	750	.99058	.00942	83	8,410	1,671	.80131	.19869
39	78.896	758	.99039	.00961	84	6,739	1,465	.78261	.21739
40	78,138	769	.99016	.00984	85	5.274	1.255	.76204	.23796
41	77.369	780	.98992	.01008	86	4,019	1,045	.73999	26001
42	76,589	795	.98962	.01038	87	2,974	844	.71621	.28379
43	75.794	809	.98933	.01067	88	2,130	659	.69061	.30939
44	74,985	827	.98897	.01103	89	1,471	495	.66349	.33651
45	74.158	847	.98858	.01142	90	976	357	.63422	.36578
46	73 311	868	.98860	.01184	91	619	245	.60420	.39580
47	72,443	892	.98769	.01231	92	374	160	.57219	.42781
48	71,551	920	.98714	.01286	93	214	99	.53738	.46262
49	70,631	948	.98658	.01342	94	115	57	.50435	.49565
50	69,683	981	.98592	.01408	95	58	31	.46552	.53448
51	68,702	1,017	.98520	.01480	96	27	16	.40741	.59259
52	67,685	1,057	.98438	.01562	97	11	7	.36364	.03636
53	66,628	1,099	.98351	.01649	98	4	3	.25000	.75000
54	65.529	1,146	.98251	.01749	99	1	1	......	1.00000

*(HUNTER'S MAKEHAMIZED.)

COMMUTATION COLUMNS. ONE LIFE. THREE AND ONE-HALF PER CENT.

Age.	D_x	N_x	S_x	M_x	R_x
10	70950	1504846	28690717	17661.57	553278.87
11	68025	1436821	27185871	17136.90	535617.30
12	65221	1371600	25749050	16632.61	518480.40
13	62532	1309068	24377450	16148.57	501847.79
14	59953	1249115	23068382	15683.99	485699.22
15	57478	1191637	21829267	15237.52	470015.23
16	55105	1136532	20627630	15808.46	454777.71
17	52829	1083703	19491098	14396.13	438969.25
18	50647	1033056	18407395	13999.90	424573.13
19	48552	984504	17374339	13618.63	410573.22
20	46545	937959	16389835	13252.26	396954.59
21	44618	893341	15451876	12899.73	383702.33
22	42770	850571	14558535	12560.53	370802.60
23	40997	809574	13707964	12223.71	358242.07
24	39296	770278	12898390	11919.26	346018.36
25	37664	732614	12128112	11616.29	334099.10
26	36099	696515	11395498	11323.97	322482.81
27	34596	661919	10698983	11041.93	311158.84
28	33154	628765	10037064	10769.81	300116.91
29	31770	596995	9408299	10507.26	289347.10
30	30441	566554	8811304	10252.88	278839.84
31	29166	537388	8244750	10007.10	268586.96
32	27942	509446	7707362	9779.30	258579.86
33	26766	482680	7197916	9538.58	248810.56
34	25638	457042	6715236	9315.04	239271.98
35	24553	432489	6258194	9097.55	229956.94
36	23511	408978	5825705	8886.26	220859.39
37	22511	386467	5416727	8680.71	211973.13
38	21549	364918	5030260	8480.22	203292.42
39	20625	344293	4665342	8284.16	194812.20
40	19736	324557	4321049	8092.71	186528.04
41	18881	305676	3996492	7905.05	178435.33
42	18058	287618	3690816	7721.14	170530.28
43	17266	270352	3403198	7540.03	162809.14
44	16504	253848	3132846	7361.97	155269.11
45	15770	238078	2878998	7186.10	147907.14
46	15063	223015	2640920	7012.07	140721.04
47	14381	208634	2417905	6839.76	133708.97
48	13724	194910	2209271	6668.67	126869.21
49	13089	181821	2014361	6498.18	120200.54
50	12477	169344	1832540	6328.44	113702.36
51	11885	157459	1663196	6158.72	107373.92
52	11313	146146	1505737	5988.74	101215.20
53	10760	135386	1359591	5818.04	95226.46
54	10225	125160.6	1224204.5	5646.56	89408.42

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(HUNTER'S MAKEHAMIZED.)

COMMUTATION COLUMNS. ONE LIFE. THREE AND ONE-HALF PER CENT.—*Cont.*

Age.	D_x	N_x	S_x	M_x	R_x
55	9706.2	115454.4	1099043.9	5473.79	83761.86
56	9203.9	106250.5	983589.5	5299.58	78288.07
57	8716.5	97534.0	877339.0	5123.52	74988.49
58	8243.7	89290.3	779805.0	4945.39	67864.97
59	7784.5	81505.8	690514.7	4765.01	62919.58
60	7338.4	74167.4	609008.9	4582.10	58154.57
61	6904.8	67262.6	534841.5	4396.67	53572.47
62	6482.9	60779.7	467578.9	4208.38	49175.80
63	6073.0	54706.7	406799.2	4017.53	44967.42
64	5674.0	49032.7	352092.5	3823.95	40949.89
65	5286.0	43746.7	303059.8	3627.83	37125.94
66	4908.6	38838.1	259313.1	3429.26	33498.11
67	4542.1	34296.0	220475.0	3228.72	30068.85
68	4186.4	30109.6	186179.0	3026.58	26840.13
69	3841.5	26268.1	156069.4	2823.26	23813.55
70	3707.8	22760.3	129801.3	2619.53	20990.29
71	3185.8	19574.5	107041.0	2416.08	18370.76
72	2875.8	16698.7	87466.5	2213.89	15954.68
73	2578.8	14119.9	70767.8	2014.07	13740.79
74	2295.2	11824.7	56647.9	1817.72	11726.72
75	2025.9	9798.8	44823.2	1626.03	9909.00
76	1771.8	8027.0	35024.4	1440.46	8282.97
77	1533.9	6403.1	26997.4	1262.44	6842.51
78	1312.8	5180.3	20504.3	1093.24	5580.07
79	1109.6	4070.65	15323.98	934.38	4486.83
80	924.68	3145.97	11253.33	787.02	3552.45
81	758.66	2387.31	8107.36	652.29	2765.43
82	611.77	1775.54	5720.05	531.04	2113.14
83	483.89	1291.65	3944.51	423.489	1582.097
84	374.64	917.01	2652.86	330.954	1158.248
85	283.28	633.73	1735.85	252.266	827.294
86	203.57	425.16	1102.12	187.138	575.028
87	149.12	276.04	676.96	134.741	387.890
88	103.19	172.852	400.924	93.854	253.149
89	68.853	103.999	228.072	63.008	159.295
90	44.139	59.860	124.073	40.622	96.287
91	27.047	32.813	64.213	25.023	55.665
92	15.789	17.024	31.400	14.6796	30.6421
93	8.729	8.295	14.376	8.1533	15.9625
94	4.532	3.763	6.081	4.2516	7.8092
95	2.209	1.554	2.318	2.0812	3.5576
96	.993	.561	.764	.9407	1.4764
97	.391	.170	.203	.3720	.5357
98	.137	.033	.033	.1316	.1637
99	.033	.000	.000	.0321	.0321

(HUNTER'S MAKEHAMIZED), FORCE OF MORTALITY.

Age.	μ_x	Age.	μ_x	Age.	μ_x	Age.	μ_x
10	.00768	33	.00863	55	.01818	77	.11531
11	.00769	34	.00875	56	.01936	78	.12729
12	.00770	35	.00888	57	.02066	79	.14060
13	.00772	36	.00902	58	.02212	80	.15540
14	.00773	37	.00918	59	.02373	81	.17183
15	.00775	38	.00935	60	.02553	82	.19010
16	.00776	39	.00955	61	.02752	83	.21040
17	.00778	40	.00977	62	.02974	84	.23295
18	.00781	41	.01001	63	.03220	85	.25801
19	.00783	42	.01028	64	.03494	86	.28586
20	.00786	43	.01058	65	.03798	87	.31681
21	.00788	44	.01091	66	.04136	88	.35120
22	.00792	45	.01128	67	.04512	89	.38941
23	.00795	46	.01169	68	.04929	90	.43187
24	.00799	47	.01215	69	.05393	91	.47905
25	.00804	48	.01265	70	.05908	92	.53149
26	.00809	49	.01321	71	.06481	93	.58975
27	.00814	50	.01384	72	.07117	94	.65449
28	.00821	51	.01453	73	.07824	95	.72643
29	.00827	52	.01531	74	.08610	96	.80637
30	.00835	53	.01617	75	.09483	97	.89521
31	.00843	54	.01712	76	.10453	98	.99392
32	.00853						

TABLE SHOWING THE ADDITION TO BE MADE TO THE YOUNGER OF TWO LIVES IN ORDER TO OBTAIN THE EQUIVALENT—EQUAL AGES

Difference of Ages.	Addition to Younger Ages.	Difference of Ages.	Addition to Younger Ages.	Difference of Ages.	Addition to Younger Ages.
1	0.513	18	12.751	35	28.660
2	1.053	19	13.626	36	29.637
3	1.618	20	14.513	37	30.616
4	2.209	21	15 410	38	31.598
5	2.826	22	16.316	39	32.581
6	3.467	23	17.230	40	33.565
7	4.132	24	18.153	41	34 552
8	4.820	25	19.083	42	35.539
9	5.530	26	20.019	43	36.528
10	6.261	27	20.961	44	37.518
11	7.013	28	21.909	45	38.509
12	7.784	29	22.862	46	39.501
13	8.572	30	23.819	47	40 493
14	9.378	31	24.781	48	41.487
15	10.200	32	25.746	49	42.481
16	11.037	33	26.714	50	43.475
17	11.887	34	27 686		

*(HUNTER'S MAKEHAMIZED.)

COMMUTATION COLUMNS. TWO LIVES. THREE AND ONE-HALF PER CENT

Equal Ages.	D_{xx}	N_{xx}	S_{xx}	M_{xx}	R_{xx}
10	7100700000	128026600000	2123131900000	2531020000	58760930000
11	6755900000	121270700000	1995105300000	2426360000	56229910000
12	6427800000	114842900000	1873834600000	2326760000	53803550000
13	6115300000	108727600000	1758991700000	2231680000	51476790000
14	5818100000	102909500000	1650264100000	2141236000	49245107000
15	5534900000	97374600000	1547354600000	2054926000	47103871000
16	5265400000	92109200000	1449980000000	1972573000	45048945000
17	5008800000	87100400000	1357870800000	1894008000	43076372000
18	4764600000	82335800000	1270770400000	1819231000	41182364000
19	4532000000	77803800000	1188434600000	1747711000	39363133000
20	4310700000	73493100000	1110630800000	1679714000	37615422000
21	4099900000	69393200000	1037137700000	1614649000	35935708000
22	3899200000	65494000000	967744500000	1552533000	34321059000
23	3707900000	61786100000	902250500000	1493153000	32768526000
24	3525900000	58260200000	840464400000	1436525000	31275373000
25	3352500000	54907700000	782204200000	1382362000	29838848000
26	3187400000	51720300000	727296500000	1330562000	28456486000
27	3029900000	48690400000	675576200000	1280908000	27125924000
28	2880000000	45810400000	626885800000	1233469000	25845016000
29	2737200000	43073200000	581075400000	1188076000	24611547000
30	2600900000	40472300000	538002200000	1144361000	23423471000
31	2471200000	38001100000	497529900000	1102571000	22279110000
32	2347500000	35653600000	459528800000	1062460000	21176539000
33	2229500000	33424100000	423875200000	1023835000	20114079000
34	2117000000	31307100000	390451100000	986764000	19090244000
35	2009700000	29297400000	359144000000	951006000	18103480000
36	1907300000	27390100000	329846600000	916515000	17152474000
37	1809600000	25580500000	302456500000	883387000	16235959000
38	1716300000	23864100000	276876000000	851214000	15352572000
39	1627200000	22237000000	253011800000	820214000	14501328000
40	1542100000	20694900000	230774800000	790107000	13681114000
41	1460800000	19234100000	210079900000	760945000	12891007000
42	1383100000	17851000000	190845800000	732651000	12130062000
43	1308700000	165,2300000	172994800000	704995000	11397411000
44	1237500000	153048 0000	156452500000	678143000	10692416000
45	1169500000	14135300000	141147700000	651965000	10014273000
46	1104300000	13031000000	127012400000	626282000	9362308000
47	1041800000	11989230000	113981380000	601189000	8736026000
48	981970000	11007260000	101992150000	576542000	8134837000
49	924530000	10082730000	90984890000	552302000	7558295000
50	869440000	9213290000	80902160000	528488000	7005993000
51	816540000	8390750000	71688870000	504995000	6477505000
52	765740000	7631010000	63292120000	481795000	5972510000
53	716940000	6914070000	55661110000	458895000	5490715000
54	670010000	6244060000	48747040000	436208000	5031820000

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(HUNTER'S MAKEHAMIZED.)

COMMUTATION COLUMNS. TWO LIVES. THREE AND ONE-HALF PER CENT.—*Cont.*

Equal Ages.	D_{xx}	N_{xx}	S_{xx}	M_{xx}	R_{xy}
55	624910000	5619150000	42502980000	413760000	4595612000
56	581570000	5037580000	36883830000	391547000	4181852000
57	539860000	4497720000	31846250000	369508000	3790305000
58	499780000	3997940000	27348530000	347684000	3420797000
59	461250000	3536690000	23350590000	326059000	3073112000
60	424250000	3112440000	19813900000	304658000	2747054000
61	388740000	2723700000	16701460000	283490000	2442396000
62	354680000	2369020000	13977760000	262576000	2158906000
63	322140000	2046880000	11608740000	242026000	1896330000
64	291040000	1755840000	9561860000	221827000	1654304000
65	261440000	1494400000	7806020000	202065000	1432477000
66	233340000	1261060000	6311620000	182796000	1230412000
67	206790000	1054270000	5050560000	164139000	1047616000
68	181810000	872460000	3996290000	146161000	883477000
69	158450000	714010000	3123830000	128940000	737316000
70	136740000	577270000	2409820000	112598000	608376000
71	116730000	460539000	1832547000	97209000	495778000
72	98453000	362086000	1372008000	82878000	398569000
73	81935000	280151000	1009922000	69689000	315691000
74	67178000	212973000	729771000	57705000	246002000
75	54171000	158802000	516798000	46969500	188296900
76	42885000	115917000	357996000	37515400	141327400
77	33265000	82652000	242079000	29345600	103812000
78	25220000	57432000	159427000	22425200	74466400
79	18646000	38786000	101995000	16704100	52041200
80	13403000	25382500	63209200	12091900	35337100
81	9338300	16044200	37826700	8480000	23245200
82	6284800	9759400	21782500	5742500	14765200
83	4069600	5689800	12023100	3739500	9022700
84	2524700	3165100	6333300	2332290	5283240
85	1494000	1671070	3168160	1386970	2950950
86	838240	832830	1497090	781770	1563980
87	443480	389350	664260	415320	782210
88	219790	169560	274910	206630	366890
89	101280	68275	105352	95548	160264
90	43079	25196	37077	40772	64716
91	16742	8454.1	11881.1	15891	23944
92	5905.1	2549.0	3427.0	5619 5	8052.6
93	1868.0	681.00	877.99	1781.8	2433 1
94	521.21	159.79	196.99	498.17	651.31
95	128.090	31.703	37.203	122.690	153.136
96	28.819	4.884	5.500	25.748	30 446
97	4.301	.583	.616	4.1359	4 6978
98	.550	.033	.033	.5298	.5619
99	.033	.000	.000	.0321	.0321

SELECT AND ULTIMATE.

$[x]$	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$	$l_{[x]+3}$	$l_{[x]+4}$	l_{x+5}
20	91690	91331	90866	90328	89716	89032
21	90968	90611	90145	89607	88996	88314
22	90246	89890	89425	88887	88277	87596
23	89528	89172	88707	88170	87561	86878
24	88808	88452	87989	87452	86842	86160
25	88089	87734	87271	86734	86125	85441
26	87370	87014	86551	86014	85404	84721
27	86650	86296	85832	85294	84684	84000
28	85933	85578	85114	84577	83965	83277
29	85212	84857	84392	83853	83240	82551
30	84491	84136	83670	83130	82514	81822
31	83768	83412	82947	82404	81786	81090
32	83041	82684	82215	81671	81051	80353
33	82317	81958	81487	80941	80316	79611
34	81585	81225	80753	80203	79574	78862
35	80850	80488	80013	79458	78822	78106
36	80111	79746	79268	78708	78067	77341
37	79364	78997	78515	77951	77302	76567
38	78612	78242	77755	77184	76528	75782
39	77852	77478	76984	76406	75741	74985
40	77081	76704	76206	75621	74945	74173
41	76301	75919	75414	74819	74131	73345
42	75509	75122	74609	74003	73301	72497
43	74707	74314	73790	73173	72454	71627
44	73884	73484	72951	72319	71581	70731
45	73045	72637	72092	71443	70684	69804
46	72186	71768	71208	70541	69754	68842
47	71302	70875	70298	69607	68791	67841
48	70390	69950	69354	68637	67789	66797
49	69445	68991	68372	67627	66742	65706
50	68462	67991	67349	66572	65648	64563
51	67440	66950	66281	65470	64502	63364
52	66374	65864	65164	64315	63299	62104
53	65257	64725	63993	63102	62035	60779
54	64088	63532	62765	61830	60709	59385
55	62861	62277	61472	60490	59310	57917
56	61572	60959	60113	59080	57839	56371
57	60213	59570	58683	57594	56287	54743
58	58783	58109	57174	56030	54655	53030
59	57277	56569	55588	54384	52937	51230
60	55692	54948	53916	52651	51133	49341
61	54022	53241	52159	50831	49238	47361
62	52266	51448	50313	48922	47253	45291
63	50420	49564	48376	46920	45177	43133
64	48484	47590	46349	44830	43015	40890
65	46458	45526	44232	42652	40767	38569
66	44341	43373	42030	40391	38441	36178
67	42137	41133	39744	38052	36047	33730
68	39843	38807	37376	35639	33596	31243
69	37494	36430	34962	33188	31108	28738
70	35072	33984	32490	30693	28602	26237

SELECT AND ULTIMATE.

[x]	$d_{[x]}$	$d_{[x]+1}$	$d_{[x]+2}$	$d_{[x]+3}$	$d_{[x]+4}$	d_{x+5}
20	359	465	538	612	684	718
21	357	466	538	611	682	713
22	356	465	538	610	681	718
23	356	465	537	609	683	718
24	356	463	537	610	682	719
25	355	463	537	609	684	720
26	356	463	537	610	683	721
27	354	464	538	610	684	723
28	355	464	537	612	688	726
29	355	465	539	613	689	729
30	355	466	540	616	692	732
31	356	465	543	618	696	737
32	357	469	544	620	698	742
33	359	471	546	625	705	749
34	360	472	550	629	712	756
35	362	475	555	636	716	765
36	365	478	560	641	726	774
37	367	482	564	649	735	785
38	370	487	571	656	746	797
39	374	494	578	665	756	812
40	377	498	585	676	772	828
41	382	505	595	688	786	848
42	387	513	606	702	804	870
43	393	524	617	719	827	896
44	400	533	632	738	850	927
45	408	545	649	759	880	962
46	418	560	667	787	912	1001
47	427	577	691	816	950	1044
48	440	596	717	848	992	1091
49	454	619	745	885	1036	1143
50	471	642	777	924	1085	1199
51	490	669	811	968	1138	1260
52	510	700	849	1016	1195	1325
53	532	732	891	1067	1256	1394
54	556	767	935	1121	1324	1468
55	584	805	982	1180	1393	1546
56	613	846	1033	1241	1468	1628
57	643	887	1089	1307	1544	1713
58	674	935	1144	1375	1625	1800
59	708	981	1204	1447	1707	1889
60	744	1032	1265	1518	1792	1980
61	781	1082	1328	1593	1877	2070
62	818	1135	1391	1669	1962	2158
63	856	1188	1456	1743	2044	2243
64	894	1241	1519	1815	2125	2321
65	932	1294	1580	1885	2198	2391
66	968	1343	1639	1950	2263	2448
67	1004	1389	1692	2005	2317	2487
68	1036	1431	1737	2043	2353	2505
69	1064	1468	1774	2080	2370	2501
70	1088	1494	1797	2051	2365	2476

SELECT AND ULTIMATE.

[x]	$p_{[x]}$	$p_{[x]+1}$	$p_{[x]+2}$	$p_{[x]+3}$	$p_{[x]+4}$	p_{x+5}
20	.996097	.994894	.994070	.993236	.992390	.991935
21	.996072	.994861	.994031	.993191	.992338	.991870
22	.996047	.994827	.993992	.993145	.992276	.991803
23	.996021	.994793	.993951	.993089	.992213	.991736
24	.995994	.994758	.993902	.993033	.992149	.991655
25	.995967	.994715	.993852	.992976	.992072	.991573
26	.995935	.994672	.993802	.992907	.991994	.991490
27	.995901	.994628	.993741	.992837	.991915	.991393
28	.995868	.994576	.993680	.992766	.991823	.991282
29	.995827	.994522	.993617	.992684	991718	.991169
30	.995786	.994468	.993545	.992590	.991611	.991054
31	.995745	.994405	.993461	.992494	.991501	.990911
32	.995696	.994333	.993377	.992396	.991365	.990766
33	.995641	.994260	.993290	.992274	.991228	.990592
34	.995584	.994185	.993183	.992151	.991062	.990414
35	.995527	.994092	.993074	.992003	.990893	.990206
36	.995455	.993998	.992944	.991852	.990696	.989992
37	.995383	.993885	.992810	.991675	.990492	.989748
38	.995296	.993769	.992654	.991493	.990261	.989483
39	.995207	.993634	.992494	.991286	.990009	.989171
40	.995103	.993495	.992311	.991061	.989712	.988837
41	.994996	.993336	.992112	.990795	.989395	.988438
42	.994874	.993164	.991878	.990511	.989016	.988000
43	.994741	.992961	.991628	.990172	.988600	.987491
44	.994585	.992744	.991328	.989800	.988116	.986894
45	.994418	.992485	.991000	.989367	.987549	.986219
46	.994219	.992200	.990618	.988860	.986908	.985459
47	.994000	.991869	.990170	.988286	.986186	.984611
48	.993745	.991481	.989664	.987640	.985380	.983667
49	.993447	.991042	.989094	.986919	.984484	.982604
50	.993109	.990548	.988458	.986117	.983474	.981429
51	.992729	.989997	.987750	.985213	.982358	.980115
52	.992305	.989384	.986953	.984215	.981109	.978665
53	.991833	.988693	.986072	.983098	.979732	.977064
54	.991302	.987929	.985086	.981865	.978211	.975280
55	.990714	.987075	.983999	.980504	.976516	.973307
56	.990057	.986132	.982798	.978988	.974642	.971120
57	.989332	.985092	.981460	.977311	.972564	.968708
58	.988532	.983932	.979980	.975452	.970273	.966057
59	.987640	.982650	.978340	.973402	.967754	.963127
60	.986653	.981228	.976531	.971148	.964971	.959871
61	.985560	.979660	.974543	.968658	.961877	.956293
62	.984354	.977937	.972345	.965890	.958478	.952353
63	.983028	.976033	.969903	.962849	.954735	.947998
64	.981563	.973916	.967220	.959500	.950598	.943238
65	.979935	.971590	.964265	.955798	.946076	.938007
66	.978146	.969029	.960998	.951752	.941107	.932335
67	.976176	.966199	.957428	.947306	.935718	.926267
68	.973999	.963105	.953505	.942685	.929954	.919822
69	.971619	.959705	.949251	.937327	.923831	.912972
70	.969003	.956018	.944700	.931849	.917323	.905629

SELECT AND ULTIMATE.

[x]	.50 $q_{[x]}$	.65 $q_{[x]+1}$	.75 $q_{[x]+2}$	.85 $q_{[x]+3}$	.95 $q_{[x]+4}$	1.00 q_{x+5}
20	.003903	.005106	.005930	.006764	.007610	.008065
21	.003928	.005139	.005969	.006809	.007662	.008130
22	.003953	.005173	.006008	.006855	.007724	.008197
23	.003979	.005207	.006049	.006911	.007787	.008264
24	.004006	.005242	.006098	.006967	.007851	.008345
25	.004033	.005285	.006148	.007024	.007928	.008427
26	.004065	.005328	.006198	.007093	.008006	.008510
27	.004099	.005372	.006259	.007163	.008085	.008607
28	.004132	.005424	.006320	.007234	.008177	.008718
29	.004173	.005478	.006383	.007316	.008282	.008831
30	.004214	.005532	.006455	.007410	.008389	.008946
31	.004255	.005595	.006539	.007506	.008499	.009089
32	.004304	.005667	.006623	.007604	.008635	.009234
33	.004359	.005740	.006710	.007726	.008772	.009408
34	.004416	.005815	.006817	.007849	.008938	.009586
35	.004473	.005908	.006926	.007997	.009107	.009794
36	.004545	.006002	.007056	.008148	.009304	.010008
37	.004617	.006115	.007190	.008325	.009508	.010252
38	.004704	.006231	.007346	.008507	.009739	.010517
39	.004793	.006366	.007506	.008714	.009991	.010829
40	.004897	.006505	.007689	.008939	.010288	.011163
41	.005004	.006664	.007888	.009205	.010605	.011562
42	.005126	.006836	.008122	.009489	.010984	.012000
43	.005259	.007039	.008372	.009828	.011400	.012509
44	.005415	.007256	.008672	.010200	.011884	.013106
45	.005582	.007515	.009000	.010633	.012451	.013781
46	.005781	.007800	.009382	.011140	.013092	.014541
47	.006000	.008131	.009830	.011714	.013814	.015389
48	.006255	.008519	.010336	.012360	.014620	.016333
49	.006553	.008958	.010906	.013081	.015516	.017396
50	.006891	.009452	.011542	.013883	.016526	.018571
51	.007271	.010003	.012250	.014787	.017642	.019885
52	.007695	.010616	.013047	.015785	.018891	.021335
53	.008167	.011307	.013928	.016902	.020268	.022936
54	.008698	.012071	.014914	.018135	.021789	.024720
55	.009286	.012925	.016001	.019496	.023484	.026693
56	.009943	.013868	.017202	.021012	.025358	.028880
57	.010668	.014908	.018540	.022689	.027436	.031292
58	.011468	.016068	.020020	.024548	.029727	.033943
59	.012360	.017350	.021660	.026598	.032246	.036873
60	.013347	.018772	.023469	.028852	.035029	.040129
61	.014440	.020340	.025457	.031342	.038123	.043707
62	.015646	.022063	.027655	.034110	.041522	.047647
63	.016972	.023967	.030097	.037151	.045265	.052002
64	.018437	.026084	.032780	.040500	.049402	.056762
65	.020065	.028410	.035735	.044202	.053924	.061993
66	.021854	.030971	.039002	.048248	.058893	.067665
67	.023824	.033801	.042572	.052694	.064282	.073733
68	.026001	.036895	.046495	.057515	.070046	.080178
69	.028381	.040295	.050749	.062673	.076169	.087028
70	.030997	.043982	.055300	.068151	.082677	.094371

SELECT AND ULTIMATE.
THREE AND ONE-HALF PER CENT.

Age. [x]	$D_{[x]}$	$D_{[x]+1}$	$D_{[x]+2}$	$D_{[x]+3}$	$D_{[x]+4}$	D_{x+5}	Age x+5
20	46080	44348	42629	40944	39292	37674	25
21	44171	42510	40861	39244	37658	36106	26
22	42339	40746	39164	37612	36091	34601	27
23	40582	39054	37536	36047	34588	33157	28
24	38894	37428	35973	34545	33144	31772	29
25	37274	35869	34473	33102	31759	30441	30
26	35720	34372	33032	31718	30428	29164	31
27	34228	32935	31650	30389	29151	27938	32
28	32797	31557	30324	29114	27926	26761	33
29	31422	30233	29050	27889	26749	25630	34
30	30102	28962	27828	26713	25619	24545	35
31	28836	27742	26654	25585	24534	23503	36
32	27618	26570	25526	24500	23491	22501	37
33	26452	25446	24144	23160	22491	21540	38
34	25330	24366	23405	22460	21530	20616	39
35	24253	23328	22406	21499	20605	19727	40
36	23219	22332	21447	20575	19718	18874	41
37	22224	21374	20525	19688	18864	18053	42
38	21270	20454	19639	18836	18044	17264	43
39	20352	19569	18787	18015	17254	16504	44
40	19469	18718	17968	17227	16495	15774	45
41	18620	17900	17180	16468	15764	15070	46
42	17804	17114	16421	15737	15061	14392	47
43	17019	16356	15692	15035	14383	13739	48
44	16262	15627	14989	14357	13730	13108	49
45	15534	14925	14312	13703	13099	12499	50
46	14832	14247	13658	13073	12490	11909	51
47	14155	13594	13027	12463	11901	11339	52
48	13501	12963	12418	11874	11331	10787	53
49	12869	12353	11828	11304	10779	10253	54
50	12258	11762	11257	10751	10243	9733.3	55
51	11667	11191	10704	10216	9724.1	9229.5	56
52	11094	10637	10168	9695.9	9220.2	8740.2	57
53	10539	10099	9647.4	9191.4	8730.5	8264.4	58
54	10000	9577.9	9142.4	8701.6	8254.9	7801.9	59
55	9476.8	9071.3	8651.3	8225.1	7792.0	7351.7	60
56	8968.5	8579.1	8173.9	7761.8	7341.8	6913.5	61
57	8474.0	8100.1	7709.6	7310.7	6903.2	6486.8	62
58	7993.0	7634.1	7257.4	6871.6	6476.3	6071.3	63
59	7524.9	7180.6	6817.4	6444.2	6060.7	5666.8	64
60	7069.2	6738.9	6388.8	6028.0	5656.1	5273.4	65
61	6625.4	6308.8	5971.6	5622.8	5262.4	4890.6	66
62	6193.3	5890.2	5565.4	5228.5	4879.4	4518.7	67
63	5772.4	5482.6	5170.2	4845.1	4507.3	4157.9	68
64	5363.2	5086.3	4786.1	4472.7	4146.5	3808.4	69
65	4965.2	4701.1	4413.1	4111.5	3796.9	3470.7	70
66	4578.8	4327.3	4051.5	3761.8	3459.2	3145.4	71
67	4204.0	3965.1	3701.6	3424.2	3134.1	2833.4	72
68	3840.7	3614.3	3363.3	3098.6	2822.1	2535.7	73
69	3492.0	3278.2	3039.8	2787.9	2524.8	2253.6	74
70	3156.0	2954.7	2729.2	2491.1	2242.8	1987.8	75

SELECT AND ULTIMATE.

THREE AND ONE-HALF PER CENT.

Age $[x]$	$N_{[x]}$	$N_{[x]+1}$	$N_{[x]+2}$	$N_{[x]+3}$	$N_{[x]+4}$	N_{x+5}	Age $x+5$
20	983407	937327	892979	850350	809406	770114	25
21	936384	892713	850203	809342	770098	732440	26
22	892286	849947	809201	770037	732425	696334	27
23	849539	808957	769903	732367	696320	661732	28
24	808559	769665	732237	696264	661719	628575	29
25	769281	732007	696138	661665	628563	596804	30
26	731633	695913	661541	628509	596791	566363	31
27	695552	661324	628389	596739	566350	537199	32
28	660980	628183	596626	566302	537188	509262	33
29	627844	596422	566189	537139	509250	482501	34
30	596095	565993	537031	509203	482490	456871	35
31	565678	536842	509100	482446	456861	432327	36
32	536529	508911	482341	456815	432315	408824	37
33	508616	482164	456718	432274	408814	386323	38
34	481874	456544	432178	408773	386313	364783	39
35	456258	432005	408677	386271	364772	344167	40
36	431731	408512	386180	364733	344158	324440	41
37	408241	386017	364643	344118	324430	305566	42
38	385756	364486	344032	324393	305557	287513	43
39	364227	343875	324306	305519	287504	270250	44
40	343622	324150	305435	287467	270240	253745	45
41	323904	305284	287384	270204	253736	237972	46
42	305039	287235	270121	253700	237963	222902	47
43	286995	269976	253620	237928	222893	208510	48
44	269736	253474	237847	222858	208501	194771	49
45	253236	237702	222777	208465	194762	181663	50
46	237465	222633	208386	194728	181655	169165	51
47	222395	208240	194646	181619	169156	157255	52
48	208003	194502	181539	169121	157247	145916	53
49	194261	181392	169039	157211	145907	135128	54
50	181147	168889	157127	145870	135119	124876	55
51	168644	156977	145786	135082	124866	115142	56
52	156728	145634	134997	124829	115133	105913	57
53	145380	134841	124742	115095	105904	97173	58
54	134585	124585	115007	105865	97163	88908	59
55	124322	114845	105774	97123	88898	81106	60
56	114581	105612	97033	88859	81097	73755	61
57	105339	96865	88765	81055	73744	66841	62
58	96587	88594	80960	73703	66831	60355	63
59	88311	80786	73605	66788	60344	54284	64
60	80497	73428	66689	60300	54272	48616	65
61	73134	66509	60200	54228	48605	43343	66
62	66209	60016	54126	48561	43332	38453	67
63	59711	53939	48456	43286	38441	33934	68
64	53631	48268	43182	38396	33923	29776	69
65	47956	42991	38290	33877	29765	25968	70
66	42676	38097	33770	29718	25956	22497	71
67	37781	33577	29612	25910	22486	19352	72
68	33257	29416	25802	22439	19340	16518	73
69	29105	25613	22335	19295	16507	13982	74
70	25303	22147	19192	16463	13972	11729	75

SELECT AND ULTIMATE.
THREE AND ONE-HALF PER CENT.

Age $[x]$	$S_{[x]}$	$S_{[x]+1}$	$S_{[x]+2}$	$S_{[x]+3}$	$S_{[x]+4}$	S_{x+5}	Age $x+5$
20	17360021	16376614	15439287	14546308	13695958	12886552	25
21	16375678	15438794	14546081	13695878	12886536	12116438	26
22	15537894	14645608	13795661	12986460	12216423	11483998	27
23	14544750	13695211	12886254	12116351	11383984	10687664	28
24	13694376	12885817	12116152	11383915	10687651	10025932	29
25	12885011	12115730	11383723	10687585	10025920	9397357	30
26	12114941	11383308	10687395	10025854	9397345	8800554	31
27	11382545	19686993	10025669	9397280	8800541	8234191	32
28	10686270	10025290	9397107	8800481	8234179	7696991	33
29	10024572	9396728	8800306	8234117	7696978	7187728	34
30	9396040	8799945	8233952	7696921	7187718	6705228	35
31	8799284	8233606	7696764	7187664	6705218	6248357	36
32	8232941	7696412	7187501	6705160	6248345	5816030	37
33	7695792	7187176	6705012	6248294	5816020	5407206	38
34	7186565	6704691	6248147	5815969	5407196	5020883	39
35	6704084	6247826	5815821	5407144	5020873	4656101	40
36	6347248	5915517	5507005	5120825	4756092	4311934	41
37	5814943	5406702	5020685	4656042	4311924	3987494	42
38	5406151	5020395	4655909	4311877	3987484	3681927	43
39	5019845	4655618	4311743	3987437	3681918	3394414	44
40	4655081	4311459	3987306	3681871	3394404	3124164	45
41	4310931	3987027	3681743	3394359	3124155	2870419	46
42	3986505	3681466	3394231	3124110	2870410	2632447	47
43	3680957	3393762	3123986	2870366	2632438	2409545	48
44	3393451	3123715	2870241	2632394	2409536	2201035	49
45	3123206	2869970	2632208	2409491	2201026	2006264	50
46	2869467	2632002	2409369	2200983	2006255	1824600	51
47	2631492	2409097	2200857	2006211	1824592	1655436	52
48	2408593	2200590	2006088	1824549	1655428	1498181	53
49	2200075	2005814	1824422	1655383	1498172	1352265	54
50	2005289	1824142	1655253	1498126	1352256	1217137	55
51	1823616	1654972	1497995	1352209	1217127	1092261	56
52	1654439	1497711	1352077	1217080	1092251	977118	57
53	1497168	1351768	1216947	1092205	977110	871206	58
54	1351238	1216653	1092068	977061	871196	774033	59
55	1216086	1091764	976919	871145	774022	685124	60
56	1091200	976619	871007	773974	685115	604018	61
57	976032	870693	773828	685063	604008	530264	62
58	870097	773510	684916	603956	530253	463422	63
59	772902	684591	603805	530200	463412	403068	64
60	683971	603474	530046	463357	403057	348785	65
61	602844	529710	463201	403001	348773	300168	66
62	529069	462860	402844	348718	300157	256825	67
63	462206	402495	348556	300100	256814	218373	68
64	401839	348208	299940	256758	218362	184439	69
65	347542	299586	256595	218305	184428	154663	70
66	298912	256236	218139	184369	154651	128695	71
67	255564	217783	184206	151594	128684	106198	72
68	217100	183843	154427	128625	106186	86846	73
69	183183	154078	128465	106130	86835	70328	74
70	153423	128120	105973	86781	70318	56346	75

SELECT AND ULTIMATE.
THREE AND A HALF PER CENT.

Age $[x]$	$M_{[x]}$	$M_{[x]+1}$	$M_{[x]+2}$	$M_{[x]+3}$	$M_{[x]+4}$	M_{x+5}
20	12824.94	12650.62	12432.47	12189.60	11920.57	11631.14
21	12489.29	12321.80	12110.57	11874.95	11616.41	11337.59
22	12165.03	12003.66	11800.01	11572.36	11322.97	11053.97
23	11853.38	11697.47	11500.71	11281.17	11040.61	10779.94
24	11551.52	11400.88	11211.59	10999.47	10766.66	10515.18
25	11260.27	11115.13	10932.24	10727.29	10502.72	10259.02
26	10978.96	10838.34	10661.63	10463.61	10246.28	10011.17
27	10706.72	10571.62	10400.52	10208.84	9998.86	9771.37
28	10444.74	10313.83	10148.52	9963.67	9760.13	9539.04
29	10190.36	10063.88	9903.81	9724.54	9527.56	9313.64
30	9944.51	9822.31	9667.32	9493.80	9302.55	9094.96
31	9706.33	9587.93	9438.51	9269.92	9084.53	8882.80
32	9475.37	9360.65	9215.04	9051.85	8872.15	8676.41
33	9252.43	9140.97	8999.68	8841.43	8666.41	8475.66
34	9034.98	8926.99	8790.19	8636.17	8465.99	8279.86
35	8824.10	8719.18	8586.17	8436.01	8269.75	8088.91
36	8619.23	8517.02	8387.69	8241.30	8079.40	7902.23
37	8419.17	8319.87	8193.87	8051.42	7893.04	7719.74
38	8224.59	8127.87	8004.87	7865.53	7710.86	7540.91
39	8034.66	7940.20	7819.65	7683.37	7531.88	7365.49
40	7848.46	7756.46	7639.04	7505.77	7356.98	7192.81
41	7666.56	7576.49	7461.45	7330.49	7184.18	7022.68
42	7488.13	7399.97	7287.06	7158.19	7013.95	6854.34
43	7313.74	7227.24	7115.59	6988.82	6846.09	6687.47
44	7140.52	7055.46	6945.95	6820.49	6678.94	6521.42
45	6970.17	6886.34	6778.15	6653.67	6513.01	6355.44
46	6801.69	6718.71	6611.30	6487.69	6340.78	6189.01
47	6634.22	6552.32	6445.39	6321.66	6180.49	6021.70
48	6467.33	6385.79	6279.07	6155.03	6013.29	5853.09
49	6300.34	6219.05	6111.96	5987.43	5844.51	5682.86
50	6132.56	6051.08	5943.77	5818.29	5674.11	5510.54
51	5964.08	5882.18	5774.14	5647.60	5501.66	5335.90
52	5794.31	5711.95	5602.73	5474.74	5326.75	5158.57
53	5622.50	5539.49	5429.14	5299.36	5149.20	4978.41
54	5448.77	5364.95	5253.23	5121.64	4969.21	4795.27
55	5272.66	5187.59	5074.30	4940.77	4785.75	4608.93
56	5093.91	5007.64	4892.60	4756.89	4599.36	4419.32
57	4911.86	4824.43	4707.90	4569.67	4409.37	4226.41
58	4726.80	4638.25	4519.57	4379.27	4216.34	4030.30
59	4538.52	4448.65	4328.34	4185.67	4020.01	3831.19
60	4347.11	4255.86	4133.57	3988.74	3820.82	3629.30
61	4152.24	4059.70	3935.82	3788.92	3618.66	3424.84
62	3954.29	3860.64	3735.09	3586.42	3414.07	3218.32
63	3753.24	3658.55	3531.58	3381.23	3207.33	3010.30
64	3549.53	3453.98	3325.83	3174.28	2999.32	2801.40
65	3343.54	3247.30	3118.20	2965.89	2790.33	2592.54
66	3135.57	3038.99	2909.53	2756.88	2581.41	2384.66
67	2926.38	2829.60	2700.23	2547.97	2373.65	2179.02
68	2716.71	2620.22	2491.45	2340.34	2168.14	1977.17
69	2507.80	2412.05	2284.41	2135.39	1966.58	1780.73
70	2300.34	2205.74	2080.24	1934.39	1779.42	1591.24

SELECT AND ULTIMATE: THREE AND ONE-HALF PER CENT.

Age $[x]$	$R_{[x]}$	$R_{[x]+1}$	$R_{[x]+2}$	$R_{[x]+3}$	$R_{[x]+4}$	R_{x+5}	Age $x+5$
20	396353.64	383528.70	370878.08	358445.61	346257.01	334336.44	25
21	383118.33	370629.04	358307.24	346196.67	334321.72	322705.31	26
22	370231.75	358066.72	346063.06	334263.05	322690.69	311367.72	27
23	357687.09	345833.71	334136.24	322635.53	311354.36	300313.75	28
24	345463.92	333912.40	322511.52	311299.93	300300.46	289533.80	29
25	333556.27	322296.00	311180.87	300248.63	289521.34	279018.62	30
26	321948.42	310969.46	300131.12	289469.49	279005.88	268759.60	31
27	310634.99	299928.27	289356.65	278956.13	268747.29	258748.43	32
28	299607.95	289163.21	278849.38	268700.86	258737.19	248977.06	33
29	288848.16	278657.80	268593.92	258690.11	248965.57	239438.01	34
30	278354.86	268410.35	258588.04	248920.72	239426.92	230124.37	35
31	268116.64	258410.31	248822.38	239383.87	230113.95	221029.42	36
32	258121.68	248646.31	239285.66	230070.62	221018.77	212146.62	37
33	248371.13	239118.70	229977.73	220978.05	212136.62	203470.21	38
34	238848.87	229813.89	220886.90	212096.71	203460.54	194994.55	39
35	229549.90	220725.80	212006.62	203420.45	194984.44	186714.69	40
36	220470.41	211851.18	203334.16	194946.47	186705.17	178625.77	41
37	211600.91	203181.74	194861.87	186668.00	178616.58	170723.54	42
38	202937.53	194712.94	186585.07	178580.20	170714.67	163003.81	43
39	194472.66	186438.00	178497.80	170678.15	162994.78	155462.90	44
40	186204.12	178355.66	170599.20	162960.16	155454.39	148097.41	45
41	178123.77	170457.21	162880.72	155419.27	148088.78	140904.60	46
42	170229.22	162741.09	155341.12	148054.06	140895.87	133881.92	47
43	162519.06	155205.32	147978.08	140862.49	133873.67	127027.58	48
44	154981.47	147840.95	140785.49	133839.54	127019.05	120340.11	49
45	147620.03	140649.86	133763.52	126985.37	120331.70	113818.69	50
46	140429.43	133627.74	126909.03	120297.73	113810.04	107463.26	51
47	133408.33	126774.11	120221.79	113776.40	107454.74	101274.25	52
48	126553.06	120085.73	113699.94	107420.87	101265.84	95252.55	53
49	119862.75	113562.41	107343.36	101231.40	95243.97	89399.46	54
50	113336.41	107203.85	101152.77	95209.00	89390.71	83716.60	55
51	106975.71	101011.63	95129.45	89355.31	83707.71	78206.05	56
52	100780.63	94986.32	89274.37	83671.64	78196.90	72870.15	57
53	94751.27	89128.77	83589.28	78160.14	72860.78	67711.58	58
54	88890.98	83442.21	78077.26	72824.03	67702.39	62733.18	59
55	83198.98	77926.32	72738.73	67664.43	62723.66	57937.91	60
56	77679.38	72585.47	67577.83	62685.23	57928.34	53328.98	61
57	72332.89	67421.03	62590.60	57888.70	53319.03	48909.66	62
58	67163.48	62436.68	57798.43	53278.86	48899.59	44683.25	63
59	62173.14	57634.62	53185.97	48857.63	44672.96	40652.95	64
60	57367.86	53020.75	48764.89	44631.32	40642.58	36821.76	65
61	52747.80	48595.56	44535.86	40600.04	36811.12	33192.46	66
62	48318.13	44363.84	40503.20	36768.11	33181.69	29767.62	67
63	44081.23	40327.99	36669.44	33137.86	29756.63	26549.30	68
64	40041.94	36492.41	33038.43	29712.60	26538.32	23539.00	69
65	36202.87	32859.33	29612.03	26493.83	23527.94	20737.61	70
66	32567.45	29431.88	26392.89	23483.36	20726.48	18145.07	71
67	29138.24	26211.86	23382.26	20682.03	18134.06	15760.41	72
68	25918.35	23201.64	20581.42	18089.07	15749.53	13581.39	73
69	22910.46	20402.66	17990.61	15706.20	13570.81	11604.23	74
70	20114.62	17814.28	15608.54	13528.30	11593.91	9823.49	75

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